Bulletin of the **S**ection of **L**ogic
Volume 9/3 (1980), pp. 102–106

reedition 2010 [original edition, pp. 102–107]

Jan Bernert

THE STRONGLY ADEQUATE MATRICES OF THE FORM OF PRODUCT OF LINDENBAUM'S MATRICES*

The paper was read at the Autumn School of Logic organized by the Section of Logic, Institute of Philosophy and Sociology, Polish Academy of Sciences, in Ustronie, September 4-12, 1979.

In this paper two problems are considered. The first one concerns existence of matrices strongly adequate for certain logics and is connected with some results presented in [6]. All matrices constructed here have the same form, more precisely, they are products of Lindenbaum's matrices. The second problem concerns Maksimova's principle of separating variables (cf. [5]).

Let $\underline{S} = \langle S, \to, +, \cdot, \sim\rangle$ be the algebra of formulas free-generated by the set $At = \{p, q, r, p_1, p_2, \ldots\}$, and let $\underline{S}_p = \langle S_p, \to, +, \cdot, \sim\rangle$ be the subalgebra generated by the set p. The algebras $\underline{S}^I = \langle S^I, \to\rangle$ and $\underline{S}^P = \langle S^P, -, +, \cdot\rangle$ are the implicational and positive fragments of the language $\underline{S}$, respectively. At (X) denotes the set of all propositional variables occurring in all formulas in X.

The symbols r_0, r_{ad} stand for the modus ponens rule and the adjunction rule, respectively. The structural consequence generated by a matrix $M = \langle A, D\rangle$ is denoted by $\overrightarrow{M}$. The symbol $\prod\limits_{t\in T} M_t$ stands for the product of matrices $\{M_t\}_{t\in T}$. We recall that $Y \in Sat(M)$ iff there exists a valuation $v : At \to A$ such that $h^v(Y) \subseteq D$. Then one can prove (cf. [4], [9]) that:

*I am grateful to Dr. P. Wojtylak for valuable suggestions. Similar paper will be published in Reports on Mathematical Logic under the title "A note on existing of matrices strongly adequate for some positive logics".

$$(*) \qquad \overrightarrow{\prod_{t \in T} M_t}(Y) = \begin{cases} \bigcap_{t \in T} \overrightarrow{M}_t(Y) & \text{if } Y \in Sat(M_t) \text{ for every } t \in T \\ S & \text{if } Y \notin Sat(M_t) \text{ for some } t \in T \end{cases}$$

We say that matrix M is strongly adequate for a structural consequence Cn iff $Cn = \overrightarrow{M}$.

Let (A) denote the following condition:

(A) $(\varphi \to \psi) \to [(\psi \to \Gamma) \to (\varphi \to \Gamma), (\psi \to \Gamma) \to \to [(\varphi \to \psi) \to (\varphi \to \Gamma)], \varphi \to \varphi \in Cn(0)$, and r_0 is derivable in Cn (i.e. $\psi \in Cn(\{\varphi, \varphi \to \psi\})$), all $\varphi, \psi, \Gamma \in S^I \quad [S^P]$,

and (B) denote the following formula:

(B) $((p \to p) \to (p \to p)) \to (p \to p) \in Cn(0)$

THEOREM 1. *For any structural consequence Cn on the language $\underline{S}^I$, if $(A), (B)$ hold for Cn, then $Cn = \overrightarrow{\prod_{X \subseteq S^I} M_X}$, where M_X is Lindenbaum's matrix of the form $M_X = \langle S^I, Cn(X), \to \rangle$.*

Now, we shall formulate an analogon of $T1$ in the language $\underline{S}^P$. To do that, let Cn be a consequence on $\underline{S}^P$ while (C) denotes the following condition:

(C) $\varphi \to \varphi + \psi, \psi \to \varphi + \psi, (\varphi \to \Gamma) \cdot (\psi \to \Gamma) \to (\varphi + \psi \to \Gamma)$, $\varphi \cdot \psi \to \varphi, \varphi \cdot \psi \to \psi, (\varphi \to \psi) \cdot (\varphi \to \Gamma) \to (\varphi \to \psi \cdot \Gamma) \in Cn(0)$ and the adjunction rule r_{ad} is derivable in Cn, all $\varphi, \psi, \Gamma \in S^P$.

THEOREM 2. *For any structural consequence Cn on the language $\underline{S}^P$, if $(A), (B), (C)$, hold for Cn, then $Cn = \overrightarrow{\prod_{X \subseteq S^P} M_X}$, $M_X = \langle S^P, Cn(X), -, +, \cdot \rangle$.*

It may be desirable to notice that the scope of $T2$ is sufficiently large. Namely, the premises of this theorem hold for the relevant logics E, R, RM, the intuitionistic logic and all their extensions (in the positive language).

What about $\underline{S}$? It is easily seen that the analogon of $T2$ need not be valid for a structural consequence on language $\underline{S}$. This can be shown by an example of a consequence Cn_E based on Belnap's and Anderson's system of entailment in its structural version (i.e. without the rule of substitution), (cf. 1).

THEOREM 3. *$Cn_E < \overrightarrow{\prod_{X \subseteq S} M_X}$, where M_X is Lindenbaum's matrix of the form $M_X = \langle S, Cn_E(X), -, +, \cdot, \sim \rangle$.*

However some consequences on $\underline{S}$ have strongly adequate matrices of the form of a product of the some Lindenbaum's matrices. Theorem 4 and 6 show it.

THEOREM 4. *Let Cn be a structural consequence on $\underline{S}$. Then, the following conditions are equivalent:*

(i) if $\varphi \in Cn(X \cup S_p)$ and $p \notin At(X \cup \{\varphi\})$ then $\varphi \in Cn(X)$; for all $\varphi \in S$, $X \subseteq S$, $p \in At$,

(ii) if $\varphi \in Cn(X \cup Y)$ and $At(Y) \cap At(X \cup \{\varphi\}) = 0$ then $\varphi \in Cn(X)$; for all $\varphi \in S, X, Y \subseteq S$,

(iii) $Cn = \overrightarrow{\prod_{X \subseteq S; p \in At} M_{X,p}}$, where $M_{X,p}$ is Lindenbaum's matrix of the form $M_{X,p} = \langle S, Cn(X \cup S_p), \to, +, \cdot, \sim \rangle$ for $p \notin At(X)$.

We write $\prod M_{X,p}$ instead of $\prod_{X \subseteq S; p \in At} M_{X,p}$.

Directly from $T4$ we get:

COROLLARY 1. *For every structural and consistent consequence Cn on $\underline{S}$ the following conditions are equivalent:*

(i) if $\varphi \in Cn(X \cup S_p)$ and $p \notin At(X \cup \{\varphi\})$ then $\varphi \in Cn(X)$, for all $\varphi \in S, X \subseteq S$, $p \in At$;

(ii) $Cn(S_p) \neq S$ for all $p \in At$ and there exists a strongly adequate matrix for Cn.

Now, let R be the set of theorems of the well-known system of relevant implication (cf. [2]) and let Cn_R be the structural consequence based on the set R and the rules: r_0, r_{ad}.

THEOREM 5. (Maksimova [5]) *Let Cn be one of the consequences Cn_E, Cn_R. If $\varphi \in Cn(X \cup \{\psi\})$ and $At(\psi) \cap At(X \cup \{\varphi\}) = 0$, then $\varphi \in Cn(X)$; for all $\varphi, \psi \in S$, $X \subseteq S$.*

It is easy to observe that the condition (i) of $T4$ and Maksimova's Theorem are equivalent for the consequences Cn_E, Cn_R. With this argument, from $T4$ and $T5$ we can immediately deduce (cf. [6]).

THEOREM 6. *If Cn is one of the consequences Cn_E, Cn_R, then $Cn = \overrightarrow{\prod M_{X,p}}$; where is such as in $T4$.*

Finally, let us examine a question similar to the previous one, that is to Maksimova's Theorem. For this purpose we shall consider only such finite structural consequences Cn on $\underline{S}^I[\underline{S}^P]$ that $Cn \leq Cn_2$, (Cn_2 is a structural version of the classical logic on $\underline{S}^I[\underline{S}^P]$). Then, by Łoś-Suszko's theorem in [3] (see also [8]) on existence of a strongly adequate matrix and by $T1$ [$T2$], we can easily deduce:

COROLLARY 2. *If $(A), (B), [(A), (B), (C)$ in $\underline{S}^P]$ hold for Cn (Cn finite, $Cn \leq Cn_2$), then for all $X, Y \subseteq S^I[S^P]$ and $\varphi \in S^I[S^P]$, $\varphi \in Cn(X)$, whenever $\varphi \in Cn(X \cup Y)$ and $At(Y) \cap At(X \cup \{\varphi\}) = 0$.*

References

[1] A. R. Anderson and N. D. Belnap, *Modalities in Ackermann's "rigorous implication"*, **Journal of Symbolic Logic** 24 (1959), pp. 107–111.

[2] N. D. Belnap, *Intensional models for first degree formulas*, **Journal of Symbolic** 32 (1967), pp. 1–22.

[3] J. Łoś, *Remarks on sentential logics*, **Indagationes Mathematicae** 20 (1958), pp. 177-183.

[4] M. Maduch, *Konsekwencja wyznaczona przez produkt kartezjański matryc*, (*Consequence operation defined by Cartesian product of matrices*, in Polish, English Summary), **Zeszyty Naukowe WSP Opole, Matematyka** XIII (1973), pp. 159–162.

[5] L. L. Maksimova, *Princip razdélńija péréménnykh v propoziciojonalnykh logikakh* (The principle of separating variables in sentential logics, in Russian), **Algebra i Logika** 15 (1976), pp. 168–184.

[6] M. Tokarz, *The existence of matrices strongly adequate for E, R and their fragments*, **Studia Logica** 38 (1979), pp. 75–85.

[7] R. Wójcicki, *Logical matrices strongly adequate for structural sentential calculi*, **Bulletin de l'Academie Polonaise des Sciences**, Serie des sciences mathematiques, astronomiques et physiques, vol. 6 (1969), pp. 333–335.

[8] R. Wójcicki, *Some remarks on the consequence operation in sentential logics*, **Fundamenta Mathematicae** 68 (1970), pp. 269–279.

[9] J. Zygmunt, *Direct products of consequence operations*, **Bulletin of the Section of Logic Polish Academy of Sciences**, vol. 1, No. 4 (1972), pp. 61–64.

Mathematical Institute
Silesian University
Katowice, Poland