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EQUIVALENTIAL LOGICS (II)

§2. Equivalential logics that have an algebraic semantics

By an *algebraic semantics* we shall mean a class $\mathbb{K}$ of matrices $\underline{M} = (\underline{A}, D)$ for a propositional language $\underline{L}$ such that D is a singleton, $D = \{d\}$. A logic $(\underline{L}, C)$ has an algebraic semantics iff $C(\emptyset) \neq \emptyset$ and there exists an algebraic semantics K strongly adequate for C, i.e., $C = Cn_{\mathbb{K}}$.

PROPOSITION (5). *If a logic $(\underline{L}, C)$ has an algebraic semantics, then every factorial matrix $\underline{M} \in Matr(C)^*$ has the following properties:*

(a) $\underline{M}$ is of the form $(\underline{A}, \{1_{\underline{A}}\})$, where $1_{\underline{A}} \in A$
(b) each formula $\alpha(p_1 \ldots p_n) \in C(\emptyset)$ defines the constant $1_{\underline{A}}$ in $\underline{A}$, that is, $\alpha_{\underline{A}}[a_1, \ldots, a_n] = 1_{\underline{A}}$ for any $a_1, \ldots, a_n \in A$.

REMARK (6). Assume that $(\underline{L}, C)$ has an algebraic semantics and let

(a) $$\underline{M} = (\underline{A}, \{1_{\underline{A}}\})$$

be in $Matr(C)^*$. $\underline{A}$ is similar to $\underline{L} = \langle L; F_1, \ldots, F_n\rangle$, say $\underline{A} = \langle A; f_1, \ldots, f_n\rangle$. To the set of operations of $\underline{A}$ we add the constant $1_{\underline{A}}$ and obtain the algebra

(b) $$\langle A; f_1, \ldots, f_n, 1_A\rangle.$$

By Proposition 5, $1_{\underline{A}}$ is definable in $\underline{A}$.

In the sequel, dealing with logics that have an algebraic semantics we do not discern between matrices (a) from $Matr(C)$ and the corresponding algebras of the form (b). This remark does not concern the congruences of (b), that is, (b) as an algebra may have many congruences but none of them (except of the identity relation) is a congruence of the matrix (a).

Given a logic $(\underline{L}, C)$ which has an algebraic semantic we define $ALG^*(C)$ to be the class of all algebras of the form (b) such that the matrix (a) belongs to $Matr(C)^*$.

Observe that each implicational logic $(\underline{L}, C)$ has an algebraic semantics. It should be mentioned that the class $ALG^*(C)$ for the classical propositional logic coincides with the class of Boolean algebras. In the case of the intuitionistic logic, $ALG^*(C)$ equals the class of pseudo-Boolean algebras.

The logic SCI of Suszko [3] is equational but it does not possess an algebraic semantics.

An equivalential logic $(\underline{L}, C)$ has an algebraic semantics iff for each $\alpha, \beta \in L$, $E(\alpha, \beta) \subseteq C(\alpha, \beta)$, where $E(p, q)$ is a C-equivalence.

Let $\mathbb{K}$ be a class of algebras closed under isomorphisms. We shall say that an algebra $\underline{A}$ is *subdirectly reducible in the class* $\mathbb{K}$ iff $\underline{A} \in \mathbb{K}$ and there exists a nonempty family $\{\Phi_\lambda : \lambda \in \Lambda\}$ of congruence of $\underline{A}$ such that

(i) for every $\lambda, \Phi_\lambda \neq \Delta_A$, where Δ_A is the diagonal of A

(ii) for every $\lambda, \underline{A}/\Phi_\lambda \in \mathbb{K}$

(iii) $\bigcap\limits_\lambda \Phi_\lambda = \Delta_A$.

Otherwise $\underline{A} \in \mathbb{K}$ is *subdirectly irreducible in* $\mathbb{K}$.

Let $(\underline{L}, C)$ be a finitistic logic. It is known that for each such logic the union of an arbitrary chain of systems from $Th(C)$ is also in $Th(C)$. Consequently, by Kuratowski-Zorn a lemma, for any $X \subseteq L$ and $\alpha \in L$ the condition $\alpha \notin C(X)$ implies the existence of a system $Y \in Th(C)$ having the properties:

(i) $\alpha \notin Y \supseteq X$

(ii) for every $Z \in Th(C)$, if $\alpha \notin Z \supseteq Y$ then $Z = Y$.

Each such $Y \in Th(C)$ is called a relatively maximal (Lindenbaum) supersystem of X with respect to α.

If $(\underline{L}, C)$ is finitistic, then $RMS(C)$ denotes the family of all relatively maximal supersystems. Thus $Y \in RMS(C)$ iff for some $X \subseteq L$ and $\alpha \notin X$, Y is a relatively maximal supersystem of X w.r. to α.

Given a logic $(\underline{L}, C)$ which has an algebraic semantics we define $\mathbb{K}_0(C)$ to be the class of all denumerable nontrivial algebras from $ALG^*(C)$ that are subdirectly irreducible in $ALG^*(C)$. Thus every algebra in $\mathbb{K}_0(C)$ has the cardinality > 1.

THEOREM (7). *Let $(\underline{L}, C)$ be a finitistic equivalential logic having an algebraic semantics. Then the following conditions are satisfied.*

(I) If $Y \in RMS(C)$ then $\underline{L}/\Theta_Y \in \mathbb{K}(C)$, where Θ_Y is the greatest congruence of the matrix $(\underline{L}, Y)$

(II) If $\underline{A} \in \mathbb{K}_0(C)$ then there exists a system $Y \in RMS(C)$ such that $\underline{A} \cong \underline{L}/\Theta_Y$.

REMARK. To be precise, the language $\underline{L} = \langle L; F_1, \ldots, F_n \rangle$ is not similar to the algebras from $ALG^*(C)$ since every algebra $\underline{A} \in ALG^*(C)$ is additionally equipped with the 0-ary operation $1_{\underline{A}}$. Writing "$\underline{L}/\Phi \in ALG(C)$" we want to indicate that there is a unique constant $1 \in L/\Phi$ definable in $\underline{L}/\Phi$ be means of an arbitrary formula from $C(\emptyset)$ such that $(\underline{L}/\Phi, \{1\}) \in Matr(C)^*$ (cf. also Remark (6).)

COROLLARY (8). *Under the assumption of Theorem (7), $\mathbb{K}_0(C)$ equals the class of all isomorphic copies of the algebras from $\{\underline{L}/\Theta_Y : Y \in RMS(C)\}$.*

Theorem (7) enables us to prove the following criterion of strong adequacy being the generalization of Theorem 1 from [4].

THEOREM (9). *Let $(\underline{L}, C)$ be a finitistic equivalential logic having an algebraic semantics. Let $\underline{B} \in ALG^*(C)$. Then the following conditions are equivalent:*

(i) $\underline{B}$ is strongly adequate for C

(ii) Every algebra from $\mathbb{K}_0(C)$ is embeddable into $\underline{B}$.

In the proof of Theorem (9) we make use of the notion of the description of an algebra (cf. [4]).

Let $(\underline{L}, C)$ be an equivalential logic w.r. to a set $E(p, q)$, having an algebraic semantics. Suppose that for a denumerable algebra $\underline{A} \in ALG^*(C)$ we are given a fixed one-to-one mapping Z from A into the propositional variables of $\underline{L}$. We define the description $DS(\underline{A})$ of the algebra $A \in ALG^*(C)$ to be the set of formulas

$\bigcup_{\gamma \in E(p,q)} \{\gamma(F(F(Z_{a_1} \ldots Z_{a_n})/p, Z_{F_{\underline{A}}(a_1,\ldots a_n)}/q) : n \in \omega,$
$F$is an n-ary connective of $\underline{L}$, $a_1, \ldots, a_n \in A\}$.

The following fact is important.

PROPOSITION (10). *Let $(\underline{L}, C)$ be an equivalential logic having an algebraic semantics. Let $\underline{A} \in ALG^*(C)$ be denumerable and let $X = C(DS(\underline{A}))$. Then $\underline{A}$ can be isomorphically embedded into the algebra $\underline{L}/\Theta_X$. (Θ_X = the greatest congruence of the matrix $(\underline{L}, X)$).*

Theorem (9) has some interesting corollaries. Some of them are placed in [1]. Here we mention the following one:

THEOREM (11). *Let $(\underline{L}, C)$ be a finitistic equivalential logic having an algebraic semantics. Assume that $C = Cn_{\underline{B}}$ for some $\underline{B} \in ALG^*(C)$. Then for each denumerable nontrivial algebra from $ALG^*(C)$ there exists an isomorphism from $\underline{A}$ into the direct power $\underline{B}^{\aleph_0}$.*

§3. Remarks on the deduction theorem

Let $(\underline{L}, C)$ be an implicational logic and let $\Rightarrow$ be its implication. We shall say that the *deduction theorem holds for* $(\underline{L}, C)$ iff for every $X \subseteq L$ and every $\alpha, \beta \in L$

$$\beta \in C(X, \alpha) \text{ iff } \alpha \Rightarrow \beta \in C(X).$$

An abstract algebra $\langle A; \Rightarrow, 1\rangle$ of type $(2, 0)$ is said to be a *positive implication algebra* ([2], p. 22) provided the following conditions are satisfied for all $a, b, c \in A$:

(p_1) $a \Rightarrow (b \Rightarrow a) = 1$
(p_2) $(a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c)) = 1$
(p_3) if $a \Rightarrow b = 1$ and $b \Rightarrow a = 1$, then $a = b$
(p_4) $a \Rightarrow 1 = 1$.

The class of all positive implication algebras is equationally definable [2].

THEOREM (12). *Let $(\underline{D}, C)$ be a finitistic implicational logic. The following conditions are equivalent:*

(a) The deduction theorem holds for C

(b) The class $ALG^*(C)$ is equationally definable, for every algebra $\underline{A} \in ALG^*(C)$, the reduct $\underline{A}^i = \langle A; \Rightarrow, 1_A \rangle$ is a positive implication algebra and for every denumerable algebra $\underline{B} \in ALG^*(C)$, $Cg(\underline{B}) = Cg(\underline{B}^i)$, i.e., every congruence of the reduct $\underline{B}^i$ is also a congruence of $\underline{B}$.

If $(\underline{L}, C)$ is implicational, then we can define a "natural" ordering in every algebra $\underline{A} \in ALG^*(C)$ putting

$$a \leq_{\underline{A}} b \text{ iff } a \Rightarrow b = 1_{\underline{A}}.$$

Then $1_{\underline{A}}$ is the greatest element in $\langle A, \leq_A \rangle$.

PROPOSITION (13). *Let $(\underline{L}, C)$ be a finitistic implicational logic. Let $\underline{A} \in ALG^*(C)$ be denumerable of power > 1. Assume that there exists greatest element in $A - \{1_{\underline{A}}\}$ with respect to $\leq_{\underline{A}}$, denoted by $*_{\underline{A}}$. Then $\underline{A} \in \mathbb{K}_0(C)$.*

PROPOSITION (14). *Let $(\underline{L}, C)$ be a finitistic implicational logic. Assume that the deduction theorem holds for C. Let $\underline{A} \in ALG^*(C)$ be denumerable of power > 1. Then the following conditions are equivalent:*

(i) $\underline{A} \in \mathbb{K}_0(C)$

(ii) There exists a greatest element $*_{\underline{A}}$ in $A - \{1_{\underline{A}}\}$ with respect to $\leq_{\underline{A}}$.

If $(\underline{L}, C)$ is the classical propositional logic, then $\mathbb{K}_0(C)$ consists of all replicas of a two-element Boolean algebra. In the case of the intuitionistic logic, $\mathbb{K}_0(C)$ consists of all denumerable strongly compact pseudo-Boolean algebras [4].

Propositions (13)–(14) together with Theorem (9) allow us easily to characterize the algebras from $\mathbb{K}_0(C)$ for some logics, for example for the n-valued Łukasiewicz logic, $n \in \omega$.

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