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ON THE LATTICE OF ELEMENTARY SITUATIONS

In [8] we adopted a set SE of *elementary situations* (E-situations) as our universe of discourse, sketching only roughly its assumed properties. These are to be specified now.

1. Prerequisites

Consider a purely conjunctive propositional language: a set of propositions L_C consisting only of the atomic ones and their conjunctions. E-situations are the *objectives* of L_C-propositions. (The objective of α – the medieval *significatum propositionis*, cf. [3] – we denote by "$S(\alpha)$".) Starting with an augmented set of E-situations SE", we assume:

1.1. SE'' comprises at least the two *improper* E-situations: the *empty* one $\underline{o}$, and the *impossible* one λ, with $o \neq \lambda$. The set $SE' = SE'' - \{\lambda\}$ is that of *possible* E-situations, and the set $SE = SE'' - \{o, \lambda\}$ of *proper* E-situations coincides with that of the *contingent* ones. (For illustrations cf. [9], [10], [11].)

1.2. SE'' is a bounded poset, with $o = zero$, and $\lambda = unit$. Thus $o \leq x \leq \lambda$, for any $x \in SE''$. The formula "$x \leq y$" is read: *x obtains in y* (cf. [5]). Two E-situations are *compossible* iff there is a proper E-situation in which they both obtain.

1.3. In SE'' any $A \subset SE''$ has a supremum. So SE'' is a complete lattice, with the join $x; y = sup\{x, y\}$, and the meet $x!y = inf\{x, y\}$. The join is to be the counterpart of conjunction. So, if $\alpha, \beta \in L_C$, then for some

$x, y \in SE'' : x = S(\alpha)$, $y = S(\beta)$, and $x; y = S(\alpha \wedge \beta)$. E-situations x, y are incompatible iff $x; y = \lambda$.

1.4. SE may be empty. But if there is any contingent E-situation at all, then there is also another one incompatible with it: $x \in SE \Rightarrow \bigvee_{y \neq \lambda} x; y = \lambda$. Thus either SE is empty, or it comprises at least two distinct and incompatible elements.

1.5. SE'' is atomistic. I.e., there is a non-empty set $SA = \{x \in SE'' : x$ covers $o\}$ of *atomic situations* ("A-situations" or "atoms"), and every E-situation is the join of some of them: $\bigwedge_{x \in SE''} \bigvee_{A \subset SA} x = supA$. Observe that $o = sup\ \emptyset$.

1.6. SE'' is distributive on condition. This is to say:

1) $(x; y \neq \lambda \ \wedge \ x; z \neq \lambda) \Rightarrow (x; y)!(x; z) \leq x; (y!z)$,
2) $(y; z \neq \lambda) \Rightarrow x!(y; z) \leq (x!y); (x!z)$.

Note the theorem: if $A \subset SA$, A is finite, $sup\ A \neq \lambda$, and $s \in SA$, then $s \leq sup\ A \Rightarrow s \in A$. Setting $At(x) = \{s \in SA : s \leq x\}$, one gets the corollary: $A = At(sup\ A)$, for any $A \subset SA$ as stipulated.

1.7. SE'' is relatively complemented on condition. I.e.,

$$(x \leq y \wedge y \neq \lambda) \Rightarrow \bigvee_{x} (x; x' = y \wedge x!x' = o).$$

Relatively to a given y, the complement of x is by 1.6.2 unique. Consequently, every interval $[o, y]$ of SE'', such that $y \neq \lambda$, is a Boolean lattice.

1.8. There is a non-empty set $SP = \{x \in SE'' : \lambda$ covers $x\}$ of maximal possible E-situations ("dual atoms", *possible worlds*, or *logical points*, cf. [5]), and every possible E-situation obtains in at least one of them: $\bigwedge_{x \in SE'} \bigvee_{w \in SP} x \leq w$. The set SP is the *logical space* of the language in question (cf. [1], [4], and [7]). Observe that $SP = \{o\}$, if SE is empty.

A corollary is: $x \leq w \Leftrightarrow x; w \neq \lambda$, for any $x \in SE''$, $w \in SP$. Hence, for any $w_i \in SP$, the set $R_i = \{x \in SE'' : x \leq w_i\}$ is a maximal ideal of SE''. Following [2], we call these ideals *realizations*, marking their totality by $\underline{R}$.

1.9. In SP, a point w_o is designated as representing the real world. Thus $R_o = \{x : x \leq w_o\}$ is the set of *real* E-situations, the rest $SE'' - R_o$ comprising all the *imaginary* ones. Clearly, $o \in R_o$.

1.10. Any two E-situations are *separated* by a possible world: $x, y \in SE''$, $x \neq y \Rightarrow \bigvee_{w \in SP} ((x \leq w \wedge \sim y \leq w) \vee (\sim x \leq w \wedge y \leq w))$. Thus, in the sense of [2], $\underline{R}$ divides SE''.

2. W-independence

An ontology satisfying 1.1 – 1.10 might be called "quasi-Wittgensteinian". To make it properly so, more has to be assumed about SA. This we precede with two definitions pertaining to SE''-sets (i.e., subsets of SE'').

2.1. Let A be an SE''-set. We call it *W-independent* (cf. [6]) iff A is not empty, *sup* $A \neq \lambda$, and for any $x, y \in A : x = y$ or $x!y = o$. Thus, in particular, $\{x, y\}$ is W-independent iff $x; y \neq \lambda$, $x!y = o$. We say then that x, y are W-independent of one another.

Observe: If A is W-independent, so is any non-empty subset of it. Save $\{\lambda\}$, every unit set is W-independent. If A is W-independent, so is $A \cup \{o\}$. The empty E-situation is W-independent of every possible one. No E-situation is W-independent of λ. If A_1, A_2 are two disjoint subsets of a W-independent SE''-set, then *sup* A_1 and *sup* A_2 are W-independent of one another.

2.2. Let $\underline{A}$ be any finite, non-empty, injectively indexed family of SE''-sets, none of them empty, and all disjoint. I.e., $\underline{A} = \{A_i\}_{i \in K}$, $K = \{1, 2, \ldots, k\}$, $A_i \subset SE''$, $A_i \neq \emptyset$, and $i \neq j$ implying $A_i \cap A_j = \emptyset$. We call that family *orthogonal* iff for any $X^k = \{x_1, x_2, \ldots, x_k\}$ such that $x_1 \in A_1$, $x_2 \in A_2, \ldots, x_k \in A_k$, the SE''-set X^k is W-independent.

So a finite family of non-empty SE''-sets is orthogonal iff it is disjoint, with every selection of E-situations, one taken from each member-set, forming a W-independent set. Observe: $\{A\}$ is orthogonal iff $A \neq \emptyset$ and $\lambda \notin A$. $\{A_1, A_2\}$ is orthogonal iff $x_1; x_2 \neq \lambda$, and $x_1!x_2 = o$, for any $x_1 \in A_1$, $x_2 \in A_2$. And $\underline{A}$ being the family of unit subsets of A, $\underline{A}$ is orthogonal iff A is W-independent.

3. Further Assumptions

Generalizing the logical atomism of [1], we stipulate:

3.1. Contingent E-situations are incompatible iff they contain incompatible atoms. Actually we have only to assume: $x, y \neq o \Rightarrow (x;y = \lambda \Rightarrow \bigvee_{s,t \in SA} (s \leq x \wedge t \leq y \wedge s;t = \lambda))$. The converse of the consequent is clearly a theorem.

3.2. Any two distinct A-situations that are incompatible with a third one, are also incompatible with each other. I.e., for any $x, y, z \in SA$, $(x;z = \lambda \wedge y;z = \lambda) \Rightarrow (x = y \vee x;y = \lambda)$. Hence the relation $x \sim_d y \Leftrightarrow (x;y \in SA \wedge (x = y \vee x;y = \lambda))$ is an equivalence on SA. The classes of the partition $\underline{D} = SA/\sim_d$ are the *logical dimensions* of L_c.

3.3. $\underline{D}$ is finite. This is arbitrary, but not quite. For arguably any human language may be of finite complexity only, and the cardinality of $\underline{D}$ might be a measure of that.

4. Some Consequences

The first is in fact the principle of logical atomism, appearing here as a theorem (by 3.1-3.3):

4.1. Provided $SE \neq \emptyset$, $\underline{D}$ is orthogonal. (If SE is empty, then $SA = \{\lambda\}$, and so $\underline{D}$ degenerates to $\{\{\lambda\}\}$.)

4.2. Every possible world contains one atom of each dimension. I.e., if $SE \neq \emptyset$, then $\bigwedge_{w \in SP} \bigwedge_{D \in \underline{D}} \bigvee_{x \in SA} (x \in D \wedge x \leq w)$. Indeed, suppose this were not the case for some $w \in SP$, $D \in \underline{D}$; i.e., we had then $x \in D \Rightarrow \sim (x \leq w)$, for every $x \in SA$. Set $\underline{a} \in D$ (D is not empty!). Hence $\sim (a \leq w)$, and by the corollary of 1.8: $a;w = \lambda$. Thus, by 3.1, $s \leq a \wedge t \leq w \wedge s;t = \lambda$, for some $s, t \in SA$. But $a \in SA$, so $s = a$, i.e. $t \leq w \wedge a;t = \lambda$. In view of the latter we have $a \sim_d t$; so $t \in D$ and $t \leq w$, contradicting the supposition.

Thus $D \cap R \neq \emptyset$, for any $D \in \underline{D}$, $R \in \underline{R}$. Clearly, $D \cap R$ is always a unit set, so in each dimension exactly one atom is real.

4.3. $Card(D) \geq 2$, for any $D \in D$ (by 1.4 and 3.1). This generalizes [1], where in effect all dimensions were minimal, i.e. such that $card(D) = 2$. Now, given any non-degenerate $\underline{D}$, we easily construct a set SA^+ corresponding to the *Sachverhalte* of [1]. Of each dimension D_i we take an arbitrary bi-partition $\{D_i^+, D_i^-\}$, members of one class to be regarded as indistinguishable. Then from each D_i^+, D_i^- we select a fixed pair of representatives $+s_i, -s_i$, taking as designated whichever belongs to the class of the atom $s_i \in R_o$. So SA^+ is simply the set of the "positive" representatives: $SA^+ = \{+s_i\}$, with $i = 1, \ldots, n$, and $n = card(\underline{D})$. Clearly SA^+ is W-independent, and $\underline{D}' = \{\{+s_1, -s_1\}, \ldots, \{+s_n, -s_n\}\}$ is orthogonal.

4.4. How are the lattices $\langle SE'', ;, ! \rangle$ and $\langle \underline{P}(SA), \cup, \cap \rangle$ related? By completeness $x = supAt(x)$, so $At(x) = At(y)$ implies $x = y$. Hence the map $\underline{At} : SE'' \to \underline{P}(SA)$ is one-one. By atomicity $At(x!y) = At(x) \cap At(y)$, so $\underline{At}$ is a meet-homomorphism into $\underline{P}(SA)$, and thus a meet-embedding. (We have also $At(x) \cup At(y) \subset At(x;y)$, but the converse fails. Take, e.g., $x, y \in D$, and $x \neq y$.)

By completeness $sup(A \cup B) = supA; supB$. So $\underline{sup} : \underline{P}(SA) \to SE''$ is a join-homomorphism and onto. (Also $sup(A \cap B) \leq supA!supB$, but the converse fails again. Take, e.g., $a \in S_1, b, b' \in D_2$, $b \neq b'$, and set $A = \{a, b\}$, $B = \{b, b'\}$.)

Finally let us note that on condition both converses hold. I.e., by corollary of 1.6: $supA$, $supB \neq \lambda \Rightarrow supA!supB \leq sup(A \cap B)$; and by 1.6(2): $x; y \neq \lambda \Rightarrow At(x;y) \subset At(x) \cup At(y)$.

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