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A REMARK ON FREE PRODUCTS

According to [2], p. 184, we admit the following

Definition. Let K be a class of similar algebras and let $\underline{A}_i$, $i \in I$, be algebras of K. Then $\underline{A}$ is the *free product in* K of the algebras $\underline{A}_i$, if:

(i) $\underline{A} \in K$
(ii) there exist $1-1$ homomorphisms ψ_i of $\underline{A}_i$ into $\underline{A}$, for $i \in I$
(iii) the set-theoretic sum $\bigcup_{i \in I} \psi_i(A_i)$ generates $\underline{A}$
(iv) if $\underline{B}$ is an algebra in K and φ_i is a homomorphism of $\underline{A}_i$ into $\underline{B}$ for $i \in I$, then there exists a homomorphism $\varphi : \underline{A} \to \underline{B}$ such that $\varphi_i = \varphi \cdot \psi_i$, all $i \in I$.

Note that φ is necessarily unique.

A free product consists of $\underline{A}$ together with the homomorphisms ψ_i, $i \in I$. If it exists, then it is unique up to isomorphism.

Our aim is to prove the following theorem:

(I) Theorem. *Let $(\underline{L}, C)$ be a finitistic equivalential logic having an algebraic semantics. Assume that $C = Cn_{\underline{B}}$ for some algebra $\underline{B} \in ALG^*(C)$. Then for each family $\underline{A}_i$, $i \in I$, of nontrivial algebras from $ALG^*(C)$, the free product of $\underline{A}_i$ in $ALG^*(C)$ exists.*

(Here we assume that $(\underline{L}, C)$ has a finite set of C-equivalences). We refer to [5] for all indispensable definitions.

The proof of theorem (I) is based on the following lemma due to D. J. Christensen and R. S. Pierce [1] (see also [2], p. 186.)

(II) Lemma. *Let K be a class of algebras such that $S(K) \subseteq K$ and $P(K) \subseteq K$. Let $\underline{A}_i \in K$, $i \in I$. The free product of the $\underline{A}_i$ exists in K*

provided that there is an algebra $\underline{C} \in K$ *and a family of embeddings* ψ_i *of* $\underline{A}_i$ *into* $\underline{C}$ *for all* $i \in I$.

$ALG^*(C)$ is closed under S and P. Thus in order to prove theorem it suffices to show the following fact:

(III) PROPOSITION. *Under the assumptions of Theorem (I), for every family* $\underline{A}_i$, $i \in I$, *of algebras from* $ALG^*(C)$ *there exists an algebra* $\underline{C} \in ALG^*(C)$ *such that each* $\underline{A}_i$ *is isomorphic with a subalgebra of* $\underline{C}$.

First we shall prove the following lemma.

(IV) LEMMA. *Let the assumptions of Theorem (I) be fulfilled. Let* $\underline{A} \in ALG(C)$ *and let* m *be an infinite cardinal such that* $2^{card(A)} \leq m$. *Then* $\underline{A}$ *is embeddable into some ultrapower* $\prod_F(\underline{B}^{\aleph_0})$ *of* $\underline{B}^{\aleph_0}$ *such that the cardinality of* $\prod_F(\underline{B}^{\aleph_0})$ *is* $\leq n^m$, *where* n *is the cardinality of the algebra* $\underline{B}^{\aleph_0}$.

PROOF. By Theorem (11) in [5] (cf. also [3]), each denumerable algebra of power > 1 from $ALG^*(C)$ is embeddable into $\underline{B}^{\aleph_0}$. Let I be the family of all nonempty denumerable subsets of A and let, for every $i \in I$, $\underline{A}_i$ be the subalgebra of $\underline{A}$ generated by the set i. By Lemma (5.5) in [4] we find an ultrafilter F over I such that $\underline{A}$ is isomorphic with a subalgebra of the reduced product $\prod_F \underline{A}_i$. But each $\underline{A}_i$ is isomorphic with a subalgebra of $\underline{B}^{\aleph_0}$. Consequently, $\prod_F \underline{A}_i$, and thus $\underline{A}$, is isomorphic with a subalegebra of the ultrapower $\prod_F(\underline{B}^{\aleph_0})$.

By a straighforward computation we get that the cardinality of $\prod_F(\underline{B}^{\aleph_0})$ is $\leq n^m$.

Since C is finitistic, $\prod_F(\underline{B}^{\aleph_0})$ is in $ALG^*(C)$. $\vdash$

Let $\mathcal{L}^*$ be the first order language describing the algebras from $ALG^*(C)$. To be precise, $\mathcal{L}^*$ has only function symbols corresponding to the operations of the algebras from $ALG(C)$. $\mathcal{L}^*$ is countable.

Now we are ready to give the proof of Proposition (III). We shall make use of some facts from the theory of models.

Let $\underline{A}_i$, $i \in I$, be a family of algebras from $ALG^*(C)$. Let m be a cardinal such that $2^{card(A_i)} \leq m$ for all $i \in I$. By Lemma (IV), for each algebra $\underline{A}_i$ we find an ultrapower $\underline{C}_i = \prod_{F_i}(\underline{B}^{\aleph_0})$ such that $\underline{A}_i$ is embedded in $\underline{C}_i$ and

(*) $\quad card(C_i) \leq m^n$ for all $i \in I$ $(n = (card(B))^{\aleph_0})$.

Each algebra $\underline{C}_i$ is a model for the countable language $\mathcal{L}^*$. Moreover, each $\underline{C}_{i_{\aleph_0}}$, being an ultrapower of $\underline{B}^{\aleph_0}$, is elementarily equivalent to $\underline{B}^{\aleph_0}$. Hence any two algebras $\underline{C}_i$, $\underline{C}_j$ $(i, j \in I)$, are elementarily equivalent.

Let T be an arbitrary set of power m^n. By Proposition 4.3.5 in [0], there exists an (m^n) – regular ultrafilter F over T. Then, by Theorem 4.3.12 [0], the ultrapower $\prod_F(\underline{B}^{\aleph_0})$ is $(m^n)^+$ – universal, that is, for every algebra $\underline{D}$ of power $\leq m^n$ and elementarily equivalent to $\prod_F(\underline{B}^{\aleph_0})$. This and (*) yield that every algebra $\underline{C}_i$ is (elementarily) embedded into $\prod_F(\underline{B}^{\aleph_0}) \in ALG^*(C)$. Thus we have shown that each algebra $\underline{A}_i$, $i \in I$, is isomorphic with a subalgebra of $\prod_F(\underline{B}^{\aleph_0})$.

Proposition (III) has been proved. This at the same time concludes the proof of Theorem (I).

The proof of Theorem (I) can be simplified in the case when the algebra $\underline{B}$ is finite.

(V) Corollary. *Let $(\underline{L}, C)$ be a finitistic implicational logic. Assume that $C = Cn_{\underline{B}}$ for some $\underline{B} \in ALG^*(C)$. Then for each family $\underline{A}_i \in ALG(C)$, $i \in I$, the free product of the $\underline{A}_i$, $i \in I$, in $ALG^*(C)$ exists.*

It would be interesting to discover topological aspects of Theorem (I).

By means of Theorem (I) one can prove the following fact.

(VI) Theorem. *Let $(\underline{L}, C)$ be a finitistic equivalential logic having an algebraic semantics. Then the following conditions are equivalent:*

(i) $(\underline{L}, C)$ has a strongly adequate matrix

(ii) The class $ALG^*(C)$ is closed under free products, that is, for any family of nontrivial algebras $\underline{A}_i \in ALG^*(C)$, $i \in I$, the free product of the $\underline{A}_i$, $i \in I$, in $ALG^*(C)$ exists.

Let $(\underline{L}, C)$ be the n-valued Łukasiewicz logic, $2 \leq n < \omega$. Then, roughly speaking, the class $ALG^*(C)$ coincides with the class of so called MV_n-algebras [6], [7]. From Theorem (I) we obtain that in (equational) class MV_n is closed under free products.

As known, the intuitionistic logic has a strongly adequate matrix. Consequently, by Theorem (I), the class of pseudo-Boolean algebras is closed under free products. But in this case, the construction of free products can be given explicitly in topological terms.

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