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THE DEGREES OF MAXIMALITY OF THE INTUITIONISTIC PROPOSITIONAL LOGIC AND OF SOME OF ITS FRAGMENTS

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Professor Ryszard Wójcicki once asked whether the degree of maximality of the consequence operation C determined by the theorems of the intuitionistic propositional logic and the detachment rule for the implication connective is equal to $2^{2^{\aleph_0}}$? The aim of the paper is to give the affirmative answer to the question. More exactly, it is proved here that the degree of maximality of C^Ψ-the Ψ-fragment of C, is equal to $2^{2^{\aleph_0}}$, for every $\Psi \subseteq \{\rightarrow, \wedge, \vee, \neg\}$ such that $\rightarrow \in \Psi$.

¿From now an Ψ stands for a subset of $\{\rightarrow, \wedge, \vee, \neg\}$ such that $\rightarrow \in \Psi$, $\underline{F}^\Psi = (F^\Psi, \Psi)$ for the absolutely free algebra determined by Ψ via infinite but denumerable set V of variables, and C^Ψ denotes the consequence operation in $\underline{F}^\Psi$ (see [4]) determined by the detachment rule for the implication connective together with all rules of the form $r(\langle \emptyset, \alpha \rangle)$ where α ranges over all theorems of the intuitionistic propositional logic which contain only connectives from the set Ψ. For $\Psi = \{\rightarrow, \wedge, \vee, \neg\}$ instead of C^Ψ we write C. Observe that the Waisberg separation theorem yields: $C^\Psi(X) = C(X) \cap F^\Psi$ for all $X \subseteq F^\Psi$.

Let $(\underline{A}, D)$ be a logical matrix (see [4]) similar to $\underline{F}^\Psi$, i.e. the algebra $\underline{A}$ is similar to $\underline{F}^\Psi$, such that $C^\Psi \leqslant C_{(\underline{A},D)}$. Set $a \equiv_D b$ iff $a \rightarrow_{\underline{A}} b$, $b \rightarrow_{\underline{A}} a \in D$ for all elements a, b of $\underline{A}$. The relation $\equiv_D$ is a congruence on $\underline{A}$. Moreover, $C_{(\underline{A},D)} = C_{(\underline{A}/\equiv_D, \{D\})}$. Since the element $\{D\}$ is definable in

$\underline{A}/_{\equiv_D}$ by the formula $x \to x$, $x \in V$, instead of $C_{(\underline{A}/_{\equiv_D},\{D\})}$ we will always write $C_{\underline{A}/_{\equiv_D}}$, and $\{D\}$ will be denoted by $1_{\underline{A}/_{\equiv_D}}$.

Put $\mathbb{K}^\Psi = \{\underline{A}/_{\equiv_D}; C^\Psi \leqslant C_{(\underline{A},D)}$ and $\underline{A}$ is similar to $\underline{F}^\Psi\}$. The domain of each algebra $\underline{A}$ from the class $\mathbb{K}^\Psi$ is partially ordered by the relation $\leqslant_{\underline{A}}$ defined as follows: $a \leqslant_{\underline{A}} b$ iff $a \to_{\underline{A}} b = 1_{\underline{A}}$ for all elements a, b of $\underline{A}$. Furthermore, the element $1_{\underline{A}}$ is the greatest element in $\underline{A}$ under the relation.

By the $\underline{A}\oplus$, where $\underline{A}$ is a member of $\mathbb{K}^\Psi$, we denote the algebra resulting from application of the well known Jaśkowski's Γ-operation (see [1]) to $\underline{A}$. Observe that $\underline{A}\oplus$ belongs to $\mathbb{K}^\Psi$ whenever $\underline{A}$ does.

Now let, $\mathbb{K}_0^\Psi$ be the subclass of $\mathbb{K}^\Psi$ consisting of all denumerable algebras of the form $\underline{A}\oplus$. It is clear that in each set $A\backslash\{1_{\underline{A}}\}, \underline{A} \in \mathbb{K}_0^\Psi$, there is a greatest element under $\leqslant_{\underline{A}}$ – this element will be denoted by $\star_{\underline{A}}$.

For each member $\underline{A}$ of $\mathbb{K}_0^\Psi$ fix some one-to-one mapping which sends each element a of $\underline{A}$ into some propositional variable Z_a of V and put

$$\begin{aligned} DS^\Psi(\underline{A}) \;=\; & \{(Z_a \otimes Z_b) \to Z_{a \otimes_{\underline{A}} b}; \otimes \in \Psi\backslash\{\neg\}, a, b \in A\} \\ & \cup\{Z_{a \otimes_{\underline{A}} b} \to (Z_a \otimes Z_b); \otimes \in \Psi\backslash\{\neg\}, a, b \in A\} \\ & \cup\{\otimes Z_a \to Z_{\otimes_{\underline{A}} a}; \otimes \in \Psi \cap \{\neg\},\ a \in A\} \\ & \cup\{Z_{\otimes_{\underline{A}} a} \to \otimes Z_a; \otimes \in \Psi \cap \{\neg\},\ a \in A\}. \end{aligned}$$

Lemma 1 (comp. A. Wroński [6]). *Let $\underline{A} \in \mathbb{K}_0^\Psi$ and $\underline{B} \in \mathbb{K}^\Psi$. Then the following conditions are equivalent:*

(i) $\underline{A}$ is embedded in $\underline{B}$

(ii) $Z_{\star_{\underline{A}}} \notin C_{\underline{B}}(DS^\Psi(\underline{A}))$.

As $\alpha_n \in F^{\{\to\}}$, $n < \omega$, take the formulae considered by A. Wroński in [7] and put $L(I) = C(Sb(\{\alpha_n; n \in I\}))$ for each $I \subseteq \omega$ (see [7]), where $Sb(X)$ denotes the closure of X under all substitutions in $F^{\{\to,\wedge,\vee,\neg\}}$.

Let $I \subseteq \omega$, and $\alpha, \beta \in F^\Psi$. Set $\alpha \equiv_I \beta$ iff $\alpha \to \beta, \beta \to \alpha \in L(I) \cap F^\Psi$. It turns out that the relation just defined is a congruence on $\underline{F}^\Psi = (F^\Psi, \Psi)$. Moreover, $\underline{F}^\Psi/_{\equiv_I} \in \mathbb{K}^\Psi$.

Let $\Re$ be some family of subsets of ω such that (i) the cardinality of $\Re$ is equal to $2^{\aleph_0}$, and (ii) if $I, J \in \Re$ and $I \neq J$, then neither $I \subseteq J$ nor $J \subseteq I$.

Lemma 2. *For all different $I, J \in \Re$ the algebras $(\underline{F}^\Psi/_{\equiv_I})\oplus$ and $(\underline{F}^\Psi/_{\equiv_J})\oplus$ are not embeddable into each other.*

Let $\mathcal{C}_F\Psi$ denote the set of all structural consequence operations in F^Ψ. $(\mathcal{C}_F\Psi, \leqslant)$ is a complete lattice (see [4]). Let Cn^Ψ denote the least upper bound in $(\mathcal{C}_F\Psi, \leqslant)$ of the set $\{C^+ \in \mathcal{C}_F\Psi; C^\Psi \leqslant C^+ \text{ and } C^\Psi(\emptyset) = C^+(\emptyset)\}$. By Makinson's result (see [2]), Cn^Ψ is structural complete in infinitary sense – the notion was originally introduced by W. A Pogorzelski in [3].

Theorem. $card\{C^+ \in \mathcal{C}_F\Psi; C^\Psi \leqslant C^+ \leqslant Cn^\Psi\} = 2^{2^{\aleph_0}}$.

Proof. The part "$\leqslant$" is obvious. Denote by $\mathcal{H}C^\Psi$, $\mathcal{H} \subseteq \mathcal{K}$, the least upper bound in $(\mathcal{C}_F\Psi, \leqslant)$ of the set $\{C^\Psi\} \cup \{C_{\{r(\langle DS^\Psi((\underline{F}^\Psi/\equiv_I \oplus), Z_{\star_{\underline{F}\Psi/\equiv_I}})\oplus\rangle)\}}; I \in \mathcal{H}\}$.

Applying Wajsberg's separation theorem, the finite approximability of $C(\emptyset)$, and Lemmas 1 and 2, one can show that the family of consequence operations just defined makes the part "$\geqslant$" hold true.

Corollary. $dmC^\Psi = 2^{2^{\aleph_0}}$.

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