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INDEPENDENT BASIS FOR THE CONSEQUENCE DETERMINED BY NON-DEGENERATED DISTRIBUTIVE LATTICES

The following rules were proved in [1] to form a basis for the consequence operation C_L defined by non-generated distributive lattices in the propositional language, with two binary connectives, $\vee$ and $\wedge$. (Any non-empty set of such lattices determines the same consequence).

1. $\dfrac{x, y}{x \wedge y}$
2. $\dfrac{x \wedge y}{x}$
3. $\dfrac{x \wedge y}{y \wedge x}$
4. $\dfrac{x}{x \vee y}$
5. $\dfrac{x \vee y}{y \vee x}$
6. $\dfrac{x \vee (y \vee z)}{(x \vee y) \vee z}$
7. $\dfrac{x \vee (y \wedge z)}{(x \vee y) \wedge (x \vee z)}$
8. $\dfrac{(x \vee y) \wedge (x \vee z)}{x \vee (y \wedge z)}$
9. $\dfrac{x \vee z}{x}$
10. $\dfrac{x \vee (x \vee y)}{x \vee y}$
11. $\dfrac{(x \vee y) \vee z}{x \vee (y \vee z)}$
12. $\dfrac{x \wedge (y \vee z)}{(x \wedge y) \vee (x \wedge z)}$

The purpose of this note is to show that: (a) rules 10-12 are redundant since they are derivable from the others, (b) 1-9 form an independent basis for C.

10 :

$$\begin{array}{ll}
x \vee (x \vee y) & \\
\hline
(x \vee y) \vee x & \text{, by 5} \\
\hline
((x \vee y) \vee x) \vee y & \text{, by 4} \\
\hline
y \vee ((x \vee y) \vee x) & \text{, by 5} \\
\hline
(y \vee (x \vee y)) \vee x & \text{, by 6} \\
\hline
x \vee (y \vee (x \vee y)) & \text{, by 5} \\
\hline
(x \vee y) \vee (x \vee y) & \text{, by 6} \\
\hline
x \vee y & \text{, by 9}
\end{array}$$

11 :
$$\begin{array}{ll}
(x \vee y) \vee z & \\ \hline
z \vee (x \vee y) & \text{, by 5} \\ \hline
(z \vee x) \vee y & \text{, by 6} \\ \hline
y \vee (z \vee x) & \text{, by 5} \\ \hline
(y \vee z) \vee x & \text{, by 6} \\ \hline
x \vee (y \vee z) & \text{, by 5}
\end{array}$$

12 :
$$\begin{array}{ll}
x \wedge (y \vee z) & \\ \hline
x, y \vee z & \text{, by 2,3} \\ \hline
x \vee z, y \vee z, x \vee (x \wedge y) & \text{, by 4} \\ \hline
z \vee x, z \vee y, x \vee (x \wedge y) & \text{, by 5} \\ \hline
(z \vee x) \wedge (z \vee y), x \vee (x \wedge y) & \text{, by 1} \\ \hline
z \vee (x \wedge y), x \vee (x \wedge y) & \text{, by 8} \\ \hline
(x \wedge y) \vee z, (x \wedge y) \vee x & \text{, by 5} \\ \hline
((x \wedge y) \vee z) \wedge ((x \wedge y) \vee x) & \text{, by 1} \\ \hline
((x \wedge y) \vee x) \wedge ((x \wedge y) \vee z) & \text{, by 3} \\ \hline
(x \wedge y) \vee (x \wedge z) & \text{, by 8}
\end{array}$$

Now we shall prove that the rules 1-9 are independent. (We say that rule r is independent of the set of rules R if and only if there is a set of formulas $X \cup \{x\}$ such that $(X, x) \in r$ and the formula x cannot be obtained from the set X using the rules in R).

Notice that the consequence C determined by rules 2-9 is such that for every set of formulas X

$$C(X) = \bigcup \{C(x) : x \in X\}, \tag{1}$$

Observe also that there are formulas x, y such that

$$x \wedge y \notin C_L(x) \cup C_L(y). \tag{2}$$

By (1) and (2) rule 1 is independent of 2-9.

For the proof of independence of the remaining rules, we use the well-known matrix method.

Let $f_1, f_2, \ldots, f_{10}$ be operations defined as follows:

f_1	0	1
0	0	1
1	1	1

f_2	0	1
0	1	1
1	1	1

f_3	0	1
0	0	0
1	1	1

f_4	0	1
0	0	0
1	0	1

f_5	0	1	2	3
0	0	3	0	3
1	3	1	2	3
2	0	0	2	3
3	3	3	3	3

f_6	0	1	2	3
0	0	2	2	0
1	2	1	2	1
2	2	2	2	2
3	0	1	2	3

f_7	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

f_8	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

f_9	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

f_{10}	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	0
3	0	1	2	3

Consider the table:

	2	3	4	5	6	7	8	9
$\vee$	f_1	f_1	f_4	f_3	f_5	f_7	f_9	f_2
$\wedge$	f_1	f_3	f_4	f_4	f_6	f_8	f_{10}	f_4
x	0	1	1	1	2	1	2	0
y	1	0	0	0	1	1	1	0
z	–	–	–	–	0	0	3	–

It indicates the interpretation of $\vee$ and $\wedge$ and the valuations which show independence of rules 2-9 of the remaining ones.

References

[1] K. Dyrda T. Prucnal, *On finitely based consequence determined by a distributive lattices*, **Bulletin of the Section of Logic**, volume 9.

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