Bulletin of the **S**ection of **L**ogic
Volume 9/3 (1980), pp. 145–147

reedition 2010 [original edition, pp. 145–149]

Marek Pałasiński

ON IDEALS IN DIRECTED COMMUTATIVE BCK-ALGEBRAS

This is an abstract of the paper presented at the seminar held by Prof. A. Wroński at the Jagiellonian University.

In [1], K. Iseki proved that every maximal ideal in an implicative BCK-algebra is prime. A. B. Thaheem established in [2] that in an implicative BCK-algebra the converse is also true. In this paper we extend the result obtained by Iseki to commutative directed BCK-algebras and we show that the converse does not hold in the general case (it holds for finite bounded commutative BCK-algebras). We give an example of a bounded commutative BCK-algebra in which a prime ideal is not maximal. The reader is referred to [3] and [4] for the definition and basic properties of BCK-algebras.

First, let us note the following lemma:

Lemma. *In a commutative direct BCK-algebra the following identity holds*

(1) $(x * y) * (y * x) = x * y$

where "directed" means that any two elements have an upper bound.

We will use this lemma to establish some properties of certain ideals in commutative directed BCK-algebras. It was proved in [5] that every bounded BCK-algebra X contains at least one maximal ideal.

Proposition 1. *Let X be a directed commutative BCK-algebra and A a maximal ideal in X. Then for all $x, y \in X$, $x * y \in A$ or $y * x \in A$.*

Proof. We have three cases

1. $x \in A$ and $y \in A$. Then $x * y \leqslant y$ imply $x * y \in A$ and $y * x \in A$.
2. $x \in A$ and $y \notin A$. Then $x * y \leqslant x$ implies $x * y \in A$.

3. $x \notin A$ and $y \notin A$. Let us suppose that $y * x \notin A$. Then, due to maximality of the ideal A, an ideal B generated by $A \cup \{y * x\}$ is equal to X. So for some natural n there are $a_0, \dots, a_n \in A \cup \{y * x\}$ such that

$$(\dots((x * y) * a_0) * \dots) * a_n = 0.$$

If all a_i, $1 \leqslant i \leqslant n$, belong to A, then $x * y \in A$. If there are a_i equal to $y * x$, then, using the fact that $(x * y) * z = (x * z) * y$ holds in any BCK-algebra and employing (1), we have

$$(\dots((x * y) * a_{i_1}) * \dots) * a_{i_k} = 0,$$

where each a_{i_j} belongs to A and therefore $x * y \in A$. This completes the proof of Proposition 1.

Proposition 2. *Let X be a directed commutative BCK-algebra and A a prime ideal in X. Then for all $x, y \in X$, $x * y \in A$ or $y * x \in A$.*

Proof. Easily follows from (1).

Theorem 1. *Let X be a directed commutative BCK-algebra and A a maximal ideal in X. Then A is a prime ideal.*

Proof. Let $x, y \in X$ and $x \wedge y \in A$. It follows from Proposition 1 that $x * y \in A$ or $y * x \in A$. Assume that $x * y \in A$. The fact that $x \wedge y \in A$ implies that three exist $a_0, \dots, a_n \in A$ such that

$$(\dots((x \wedge y) * a_0) * \dots) * a_n = 0.$$

But in a commutative BCK-algebra $x \wedge y = x * (x * y)$. Thus we have

$$(\dots((x * (x * y)) * a_0) * \dots) * a_n = 0$$

and this equality means that $x \in A$.

In the same manner we can prove that if $y * x \in A$ then $y \in A$. This completes the proof of Theorem 1.

Let us recall two following Theorems.

Theorem 2. (see [4]). *Let X and Y be BCK-algebras. A non-empty subset I of the product $X \times Y$ is an ideal in $X \times Y$ iff $I = I_1 \times I_2$, where I_1, I_2 are ideals of X and Y, respectively.*

Theorem 3. (see [6]). *Every finite bounded commutative BCK-algebra is a product of linearly ordered BCK-algebras.*

It was noticed in [6] that any finite linearly ordered commutative BCK-algebra is simple. Thus it is easily seen, using Theorem 2, that in the finite bounded commutative BCK-algebra each prime ideal is maximal. In the general case however this does not hold true. We have the following counterexample.

Example: Let $X = \{x : x \in \mathbb{R}, 0 \leqslant x \leqslant 1\}$. We define a binary operation on X by $x * y = max\{0, x - y\}$. $\langle X, *, 0\rangle$ is a commutative bounded lineary ordered BCK-algebra (see [3]).

Let F be an ultrafilter over $\mathbb{N}$ generated by cofinite subsets of $\mathbb{N}$. We form an ultrapower $X^{\mathbb{N}}/_F$ of X and denote it by X_F. Clearly it is a commutative bounded lineary ordered BCK-algebra. Let I denote the set of all infinitesimals in X_F (see [7]) and 0_F the smallest element of X_F. It can be proved that $I \neq \emptyset$ and $I \cup \{0_F\}$ forms a non-trivial ideal in X_F. This ideal is a maximal ideal in X_F. The ideal $\{0_F\}$ is prime but it is not maximal in X_F because $\{0_F\} \subsetneq I \cup \{0_F\}$.

References

[1] K. Iseki, *On some ideals in BCK-algebras*, **Mathematics Seminar Notes** 3 (1975), pp. 65–70.

[2] A. B. Thaheem, *Characterizations of certain ideals in implicative BCK-algebras*, **Mathematics Seminar Notes** 6 (1978), pp. 465–481.

[3] K. Iseki, S. Tanaka, *An introduction to the theory of BCK-algebras*, **Mathematics Japonica** 23 (1978), pp. 1–26.

[4] K. Iseki, S. Tanaka, *Ideal theory of BCK-algebras*, **Mathematica Japonica** 21 (1976), pp. 351–366.

[5] J. Ahsan, A. B. Thaheem, *On ideals in BCK-algebras*, **Mathematics Seminar Notes** 5 (1977), pp. 167–172.

[6] A. Romanowska, T. Traczyk, *On commutative BCK-algebras*, preprint.

[7] A. Robinson, **Non-standard analysis**, North Holland Publishing Company 1974.

Mathematical Institute
Jagiellonian University
Cracow