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EVERY TWO-VALUED PROPOSITIONAL CALCULUS HAS THE INTERPOLATION PROPERTY

It is known that two-valued calculi, one with implication, negation (+ other connectives) and the other pure implicational (cf. [1]) have the Interpolation Property. In this paper we prove that every two-valued calculus with implication + other connectives has this property.

Let $\underline{L} = (L, Con)$ be an algebra of formulas formed in the usual manner by means of propositional variables and operations from Con denoted by propositional connectives. We assume that Con contains the implication ($\rightarrow$) and that all operations of Con are finite. Let $\underline{M}$ be a matrix (connected with the language $\underline{L}$) with the set $\{0, 1\}$ as the universum and 1 as the distinguished value. We also assume that $\rightarrow$ is defined in $\underline{M}$ in the usual manner. Symbols V_a $(a \in L)$ and T denote the set of variables of the formula a and the set of tautologies of the matrix $\underline{M}$, respectively.

Theorem. *If $a \rightarrow b \in T$ and $V_a \cap V_b \neq \emptyset$, then there exists a formula c such that $V_c \subseteq V_a \cap V_b$ and $a \rightarrow c$, $c \rightarrow b \in T$.*

Consider two cases: 1^o. there exists an n argument $(n \geqslant 0)$ connective f of Con such that $f(1, \underbrace{\ldots}_{n}, 1) = 0$, 2^o. such a connective does not exist.

If 1^o holds, then the formula: $p \rightarrow f(p \rightarrow p, \ldots, p \rightarrow p)$ defines the classical negation, and in this case the proof is well-known. Assume that the second case holds. Denote the set of formulas $V_a \cap V_b \cup \{p \rightarrow p : p \in V_a \cap V_b\}$ by the symbol W.

Let S be the set of all substitutions s such that for every variable p the following conditions are satisfied:

if $p \in V_a - V_b$ then $sp \in W$,

if $p \notin V_a - V_b$ then $sp = p$.

Since the set $V_a - V_b$ is finite, then the set S is also finite. So, let $c = s_1 a \vee \ldots \vee a_k a$, where $\{s_1, \ldots, s_k\} = S$. (The symbol $\vee$ denotes the classical disjunction. This connective is obviously defined by $\rightarrow$). If $s \in S$, then $s(a \rightarrow b) = sa \rightarrow b$. Hence $c \rightarrow b \in T$. Now, we prove that $a \rightarrow c \in T$. Suppose that for some valuation v $va = 1$ and $vc = 0$. Hence and by assumption 2^o, there is a variable $p \in V_a \cap V_b$ such that $vp = 0$. Let s be a substitution such that for every variable q:

$$\text{if } q \in V_a - V_b \text{ then } sq = \begin{cases} p & \text{if } \quad vq = 0 \\ p \rightarrow p & \text{otherwise} \end{cases}$$

if $q \notin V_a - V_b$ then $sq = q$.

So for every variable q, $vq = vsq$. Hence $va = vsa$. It can easily be noticed that $s \in S$. So $1 = vsa \leqslant vc$. Contradiction.

References

[1] R. Edelstein, *An interpolation lemma for the pure implicational calculus*, **Journal of Symbolic Logic** 40 (1975), pp. 443–444.

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