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QUASI EQUATIONAL LOGIC OF PARTIAL ALGEBRAS

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In the line of introducing a manageable model theoretic approach to partial algebras, here such classes of partial algebras are to be considered in which free algebras still exist (in a categorical language: which are epireflective). This note is to be understood as one among others introducing this kind of model theory (in another one, see [4], varieties of partial algebras are considered). We want to make an end to the widely spread opinion that there are several equational theories and several notions of validity around for partial algebras. At the same time we want to provide for all those who might use partial algebras such tools that they hopefully can really work with.

We just use the usual first order formulas of a model theoretic language with terms but substituting the notion of "equation" by that of "existence-equation", and we intend to give a procedure how to interpret their satisfaction and their validity in partial algebras. In this note we shall restrict ourselves to existence-equations and quasi-existence-equations (the latter comparable to the notion of quasi-equation in [8] §11.1., i.e. essentially: universally quantified Horn-formulas).

Motivation: Any approach to an "equational theory" for partial algebras, tried so far, gets into some conflict with the usual feeling of equality, developed by dealing with full algebras, where one does not have to care about the "existence" of the values of the terms involved. But when saying something about the satisfaction or the validity of what one would like to be an "equation" for partial algebras one always includes some sort of non-trivial statement about the existence of the values of the terms involved. This becomes clear, especially, when one wants to deal also with negations (see [4] for the general context). Thus we have decided to replace the notion of "equation" by that of "existence-equation" (briefly E-equation), written as "$t \stackrel{e}{=} t'$", to make the reader aware that something is really different when dealing with "equality" in connection with partial algebras.

Intuitive definitions: Intuitively spoken, an *existence-equation* $t \stackrel{e}{=} t'$ consists of a pair of terms, t and t', and a *quasi-existence-equation* (briefly, a QE-equation) is an implication $t_1 \stackrel{e}{=} t'_1 \wedge t_2 \stackrel{e}{=} t'_2 \wedge \ldots \wedge t_n \stackrel{e}{=} t'_n \rightarrow t_0 \stackrel{e}{=} t'_0$, where the premise consists of a conjunction of E-equation and the conclusion is a single E-equation. Note that in this context an E-equation can always be viewed at as a QE-equation with an empty premise.

Following the usual procedure, we say that an E-equation φ, or a QE-equation φ respectively, is valid in a partial algebra $\underline{A}$ (in short $\underline{A} \models \varphi$) iff φ is satisfied in $\underline{A}$ by every evaluation $k : V \to A$ of the set V of variables (note: "iff" stands for "if and only if"). Thus we have only to make precise what we mean by satisfaction. Since in a partial algebra $\underline{A}$ the value of a term t does not always exist under an evaluation k of the variables, one has to be especially careful at this place. We take here the definition which has so far been used in connection with the so called "strong validity of equations" (see [1]-[3], [5]-[7]). We say that an E-equation $t \stackrel{e}{=} t'$ is *satisfied in a partial algebra* $\underline{A}$ *by an evaluation* $k : V \to A$ of the set V of variables (in short: $\underline{A} \models t \stackrel{e}{=} t'[k]$), iff both the values of t and t' exist and these two values coincide. Thus, if we state that an E-equation $t \stackrel{e}{=} t'$ is satisfied in $\underline{A}$ under an evaluation k we always include the claim that the values of the terms involved to really exist in $\underline{A}$ under k.

Therefore the *negation* of the statement "$\underline{A} \models t \stackrel{e}{=} t'[k]$", i.e. "$\underline{A} \not\models t \stackrel{e}{=} t'[k]$" may mean one of the followings: *Either* at least one of the values of the terms t and t' does not exist in $\underline{A}$ under k, *or* both values exist but they do not coincide.

What does it mean that "the value of a term t exists in the partial algebra $\underline{A}$ under the evaluation K^*? This is defined by induction:
The value of a term t in the partial algebra $\underline{A}$ under the evaluation k is denoted by $t^{\underline{A}}$ under the evaluation k is denoted by $t^{\underline{A}}[k]$.

(i) If $v \in V$ is a variable then $v^{\underline{A}}[k]$ exists, and $v^{\underline{A}}[k] = k(v)$.

(ii) If f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms about which we already know that $t_1^{\underline{A}}[k], \ldots, t_n^{\underline{A}}[k]$ exist, then $ft_1 \ldots t_n^{\underline{A}}[k]$ exists iff the fundamental operation $f^{\underline{A}}$ of $\underline{A}$ corresponding to f is defined on the sequence $(t_1^{\underline{A}}[k], \ldots, t_n^{\underline{A}}[k])$. Then $ft_1 \ldots t_n^{\underline{A}}[k] = f^{\underline{A}}(t_1^{\underline{A}}[k], \ldots, t_n^{\underline{A}}[k])$.

Now it is easy to define the *satisfaction of a QE-equation* $\varphi := (t_1 \stackrel{e}{=} t'_1 \wedge \ldots \wedge t_n \stackrel{e}{=} t'_n \rightarrow t_0 \stackrel{e}{=} t'_0)$ in a partial algebra $\underline{A}$ under an evaluation k: The statement "$\underline{A} \models \varphi[k]$" is true iff whenever k satisfies all the E-equations $t_i \stackrel{e}{=} t'_i$ for $1 \leq i \leq n$, then k also satisfies the E-equation $t_0 \stackrel{e}{=} t'_0$.

Note that every existence-equation or implication will always be valid in the empty algebra $\underline{0}$ with empty carrier set and empty partial operations, since by lack of evaluations *every* evaluation satisfies the equation or implication.

The main results: Let $\underline{P}^{\Delta}$ be the class of all partial algebras of some given type Δ, let $\underline{K} \subseteq \underline{P}^{\Delta}$ be any subclass. Let $QEeq\underline{K}$ denote the class of all QE-equations with terms corresponding to the type Δ which are valid in $\underline{K}$ and let $Mod\ QEeq\underline{K}$ denote the class of all those partial algebras in $\underline{P}^{\Delta}$ in which every QE-equation of $QEeq\underline{K}$ is valid. Moreover let $0\underline{K} := \underline{K} \cup \{\underline{0}\}$, let $P^r\underline{K}$ denote the class of all reduced products of $\underline{K}$-algebras defined along the line as one defines reduced products for relational structures, see [8], let $S\underline{K}$ denote the class of all subalgebras of $\underline{K}$-algebras (i.e. the carrier set is a "closed subset" and the operations are the induced ones) and let $I\underline{K}$ denote the class of all $\underline{P}^{\Delta}$-algebras isomorphic to some $\underline{K}$-algebras.

THEOREM 1. *Let $\underline{K} \subseteq \underline{P}^{\Delta}$ be an arbitrary class of partial algebras of type Δ, then*

$$Mod\ EEeq\underline{K} = ISP^r 0\underline{K}.$$

Thus especially $Mod\ QEeq\underline{K}$ is always closed with respect to subalgebras and direct products, and therefore free algebras exist in this class for every set (and universal solutions for every $\underline{P}^{\Delta}$-algebra, see [9]), i.e. in a categorical language, $Mod\ QEeq\underline{K}$ is epireflective. (The characterization of $Mod\ Eeq\underline{K}$ is contained in [4].)

The proof of Theorem 1 can be given in Zermelo-Fraenkel set theory without the Axiom of Choice (in short: AC), i.e. ZF-set-theory. Thus we have

THEOREM 2. *In ZF-set-theory, i.e. without AC, there holds for every class* $\underline{K} \subseteq \underline{P}^{\Delta}$ *that*

$$Mod\ QEeq\underline{K} = ISP^{c}0\underline{K},$$

where the operator P^r *is replaced by the operator* P^c *(see below).*

Here $P^c\underline{K}$ denotes the class of all reduced products of $\underline{K}$-algebras in the "categorical sense". I.e. if $(A_i \mid i \in I)$ is a family of partial algebras and F is any filter on I, then this reduced product $P^c_{i\in I}\underline{A}_i/F$ is defined to be the direct limit of the directed system $(\underline{A}_X \xrightarrow{\varphi_{XY}} \underline{A}_Y \mid X, Y \in F,$ $X \subseteq Y)$, where $\underline{A}_X := P_{i\in X}\underline{A}_i$ denotes the Cartesian product of the family $(\underline{A}_i \mid i \in X)$, and φ_{XY} is the "restriction mapping" reducing X-sequences to Y-sequences.

We denote a reduced product which is defined in the set theoretical way as indicated earlier by $P^r_{i\in I}\underline{A}_i/F$, and we consider the statement

(∗) For every set I, for every family $(\underline{A}_i \mid i \in I)$ of *nonempty* partial algebras of type Δ, and for every (proper) filter F on I there always holds $P^r_{i\in I}\underline{A}_i/F \simeq P^c_{i\in I}\underline{A}_i/F$.

THEOREM 3. *ZF-set-theory (i.e. without AC) the Axiom of Choice and statement (∗) are equivalent:*

(i) (ZF+AC) implies (∗) and

(ii) (ZF)+(∗)) implies AC.

The paper also include an algebraic characterization of those sets of QE-equation closed under consequences. This is a generalization of Birkhoff's completeness theorem for equations in total algebras.

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