Bulletin of the **S**ection of **L**ogic
Volume 9/4 (1980), pp. 154–157
reedition 2010 [original edition, pp. 154–158]

Zdzisław Dywan

QUASI-STRONGLY FINITE SENTENTIAL CALCULI

Let $\underline{L} = (L, Con)$ be an absolutely free algebra generated by the finite set of generators $\{p_1, p_2, \ldots\}$ (the set of sentential variables), where Con is a finite sequence of (non-nullary) operations denoted by sentential connectives. $\underline{L}_k = (L_k, Con)$ is the subalgebra of $\underline{L}$ generated by variables $p_1, \ldots, p_k$. The rules (finite or and infinite) are defined in the usual manner. Sb is the consequence determined by the rule of substitution. If C is a consequence operation of $\underline{L}$, R a set of rules, M a generalized matrix (cf. [7,6]), then $\overline{C}$, C_R, Cn_R, Cn_M, $\approx_M$ denote respectively: $C = Sb$, the consequence obtained by adding the set of rules R, to the rules of C, the consequence determined by the rules of R, the consequence determined by the matrix M, the relation such that for all formulas a, b, $a \approx_M b$ if and only if for every valuation in M $va = vb$. A substitution $e : \underline{L} \to \underline{L}_k$ is called a k-substitution. We often write M instead of Cn_M. Some of the notions not defined in this paper can be found in [7,6].

A (quasi) structural consequence C is (quasi) strongly finite if there is a finite generalized matrix M such that $C = \overline{M}$) $C = M$. G_k is a rule of the form:

$$\frac{a(p_{n_i}/p_{n_j}) : 1 \leqslant i \neq j \leqslant k+1}{a}$$

where $p_{n_1}, \ldots, p_{n_{k+1}}$ are any variables and a any formula. This rule is similar to Wroński's rule r_n in [8]. We say that $\approx$ is a C-congruence of $\underline{L}$ if for all formulas a, b: if $a \approx b$ then $C(\{a\}) = C(\{b\})$.

Theorem 1. *A quasi-structural consequence C is quasi-strongly finite if and only if there is a natural number k such that G_k is a rule of C and there is a C-congruence $\approx$ such that the set $L_k/_{\approx}$ is finite.*

We say that the consequence C_1 is a structural strengthening of C if there is a set of structural rules R such that $C_1 = C_R$.

THEOREM 2. *Every structural strengthening of a quasi-strongly finite consequence is also quasi-strongly finite and can be obtained by a finite set of standard rules.*

THEOREM 3. *If M is a finite matrix, then there is a decision procedure which enables us to construct:*

(i) for any finite set of standard rules R, a finite matrix N such that $\overline{N} = \overline{M}_R$,

(ii) the lattice of all structural strengthening of $\overline{M}$.

Let r be a rule and X any set of formulas. $r \in Perm(X)$ if and only if for every formula a, the set of formulas Y and the substitution e, if $(Y, a) \in r$ then

$$\text{if } eY \subseteq X \text{ then } ea \in X.$$

Let C be a consequence. $r \in Der(C)$ if and only if $C_{\{r\}} \leqslant C$. A consequence C is structurally complete in the infinite sense if $Perm(C(\emptyset)) \cap Struct \subseteq Der(C)$, where $Struct$ denotes the set of all structural rules (cf. [3,4]).

Let K be a class of elementary logical matrices. By $S(K)$ we denote the least class of matrices containing K and closed under the operation of forming submatrices. If $M = (\underline{A}, \{A_i : i \subseteq I\})$ is a generalized matrix ($\underline{A} = (A, Con)$ is an algebra similar to $\underline{L}$ and for every $i \in I$, $A_i \subseteq A$), then M^o denotes the set of all elementary matrices $\{(\underline{A}, A_i) : i \in I\}$.

THEOREM 4. *A consequence $\overline{M}$ is structurally complete if and only if $Perm(M(\emptyset)) \cap Struct \subseteq \bigcap\{Perm(N(\emptyset)) : N \in S(M^o)\}$.*

THEOREM 5. *There is a quasi-structural consequence such that it is a maximal non-quasi-strongly finite consequence and a maximal consequence with the degree of maximality (completeness) equal to $\aleph_0$.*

COROLLARY 1. *Quasi-strongly finite consequences form the non-complete lattice.*

COROLLARY 2. *There is no minimal (quasi) strongly finite consequence.*

COROLLARY 3. *There is a quasi-strongly finite consequence whose degrees of completeness and of maximality are not equal.*

G. Malinowski conjectured (cf. [1]) that this corollary does not hold.

A consequence C is (quasi) finitely based if and only if there is a finite set of standard rules R such that $C = Cn_R$ ($C = \overline{Cn_R}$). In [5], P. Wojtylak proved that there is a strongly finite consequence C_W, in the language $\underline{L}$ with only one binary connective, such that for every finite set of standard rules R, $C_W(\emptyset) \neq Cn_R(\emptyset)$.

COROLLARY 4. *Every quasi strongly finite consequence $C \leqslant C_W$ is not (quasi) finitely based.*

References

[1] G. Malinowski, **Topics in the theory of strengthening of sentential calculi**, The Institute of Philosophy and Sociology of the Polish Academy of Sciences, Warszawa 1979.

[2] W. A. Pogorzelski, *Structural completeness of the propositional calculus*, **Bulletin de l'Académe Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques**, vol. XIX, No. 5 (1971), pp. 349–351.

[3] W. A. Pogorzelski and T. Prucnal, *Equivalence of the structural completeness theorem for propositional calculus and the Boole'an representation theorem*, **Reports on Mathematical Logic**, No. 3 (1974), pp. 37–40.

[4] T. Prucnal, *On the structural completeness of some pure implicational propositional calculi*, **Studia Logica** XXX (1972), pp. 45–52.

[5] P. Wojtylak, *Strongly finite logics: Finite axiomatizability and the problem of supremum*, **Bulletin of the Section of Logic**, Polish Academy of Sciences, Vol. 8, No. 2 (1979), pp. 99-111.

[6] R. Wójcicki, *Matrix approach in methodology of sentential calculi*, **Studia Logica** XXXIII (1973), pp. 7–39.

[7] R. Wójcicki, **Strongly finite sentential calculi: Selected papers on Łukasiewicz sentential calculi**, Eds. R. Wójcicki and G. Malinowski, Warszawa-Wrocław 1976.

[8] A. Wroński, *On the depth of a consequence operation*, **Bulletin of the Section of Logic**, Polish Academy of Sciences, Vol. 6, No. 3 (1977), pp. 96–101.

The Catholic University of Lublin