Bulletin of the **S**ection of **L**ogic
Volume 9/4 (1980), pp. 163–165

reedition 2010 [original edition, pp. 163–165]

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AN EXAMPLE OF THE COMMUTATIVE BCK-ALGEBRA

A. Romanowska and T. Traczyk posed the following problem (see [1] Problem 1):

Does there exist a commutative BCK-chain that is not subdirectly irreducible?

In this note we shall solve the above problem in the affirmative by constructing a linearly ordered commutative BCK-algebra which is not subdirectly irreducible. For the definition and properties of BCK-algebras we refer the reader to [2] and [3]. We will make use of an ultraproduct construction. For details concerning this concept see [4].

It was shown in [3] that the algebra $X = \langle\{x : x \in \mathbb{R}, 0 \leqslant x \leqslant 1\}, *, 0\rangle$ where $*$ is defined by $x*y = max\{0, x-y\}$ is a commutative BCK-algebra. Let F be an ultrafilter over $\mathbb{N}$ generated by cofinite subsets of N. Let us consider an ultrapower $X^{\mathbb{N}}/_F$ of X and denote it by X_F. We will denote elements of X_F by $\overline{x}, \overline{y}, \ldots$. It is easy to see that X_F is an elementary extension of X and so we have an elementary embedding $f : X \to X_F$. Every $\overline{x}$ such that for some $x \in X$, $\overline{x} = f(x)$ will be called a real number. Clearly X_F is a commutative BCK-algebra linearly ordered by relation $\leqslant_F$. Let us denote the binary operation and the zero element of the algebra X_F by $*_F$ and 0_F, respectively. We have the following definition (see [4]).

Definition. A non-zero element $\overline{y} \in X_F$ is called infinitesimal iff for every non-zero real number $\overline{x} \in X_F$, $\overline{y} \leqslant_F \overline{x}$.

It is easily seen that the set of all infinitesimal elements is not empty.

Let us recall the following Lemma:

Lemma 1. *Let $Y = \langle Y, *, 0\rangle$ be a BCK-algebra. An element $y \in Y$ belongs* *to an ideal generated by x iff for some natural number k*

$$(\ldots(y * \underbrace{x) * \ldots) * x}_{k-times} = 0$$

To prove that the algebra X_F is not subdirectly irreducible it is sufficient to show the following:

LEMMA 2. *For every infinitesimal $\overline{x} \in X_F$ there exists a nontrivial ideal I such that $\overline{x} \notin I$.*

PROOF. Let $\overline{x}$ be an infinitesimal element determined by a sequence $(x_0, x_1, \ldots)$. Then for any $x \in (0,1]$ $\{i : 0 < x_i \leqslant x\} \in F$. Let $\overline{x}$ be determined by a sequence $(x'_0, x'_1, \ldots, x'_n, \ldots) = (x_0, x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n, \ldots)$. It is obvious that $\{i : 0 < x'_i < x_i\} \in F$, so $\overline{x}' <_F \overline{x}$. We will prove that $\overline{x}$ does not belong to the ideal generated by $\overline{x}'$. Following Lemma 1 it is sufficient to prove that for any natural k an element $\overline{z}^{(k)}$ defined by

$$\overline{z}^{(k)} = (\ldots(\overline{x} \underbrace{*_F \overline{x}') *_F \ldots) *_F \overline{x}'}_{k-times}$$

is not equal to 0_F.

Let us observe that $\overline{z}^{(k)}$ is determined by a sequence $(z_0^{(k)}, z_1^{(k)}, \ldots)$ where $\overline{z}_n^{(k)} = \max\{0, x_n - \frac{k}{n}x_n\}$. It is easy to see that for $n > k$ $x_n > 0$ iff $z_n^{(n)} > 0$. Hence the set

$$\{i : x_i > 0\} - \{i : z_i^{(k)} > 0\}$$

is finite and $\{i : x_i > 0\} \in F$ implies $\{i : z_i^{(k)} > 0\} \in F$. Thus we have proved that for any natural k $\overline{z}^{(k)} >_F 0_F$ and therefore $\overline{x}$ does not belong to the ideal generated by $\overline{x}'$. This completes the proof of Lemma 2.

References

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