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FINITELY GENERATED IDEALS IN DIRECTED COMMUTATIVE BCK-ALGEBRA

This main aim of this paper is to prove that in a direct commutative BCK-algebra an ideal I is finitely generated if and only if I is a principal ideal. This result generalizes the result obtained by E. Y. Deeba in [2]. We also give an answer to the question posed by E. Y. Deeba in [1]: for what class of BCK-algebras is every Noetherian algebra a principal ideal algebra (i.e. an algebra whose all ideals are principal)?

By a BCK-algebra we mean a general algebra $X = \langle , *, 0\rangle$ of type $\langle 2, 0\rangle$ satisfying the following conditions:

I. $(x * y) * (x * z) \leqslant z * y$
II. $x * (x * y) \leqslant y$
III. $x \leqslant x$
IV. $0 \leqslant x$
V. $x \leqslant y$, $y \leqslant x \Rightarrow x = y$

where $x \leqslant y$ means $x * y = 0$.

It is easily seen that $\leqslant$ is a partial ordering on X (see for example [4]).

A BCK-algebra X is called commutative iff $x \wedge y = y \wedge x$, where $x \wedge y$ is defined as $y * (y * x)$.

It was shown in [4] that a commutative BCK-algebra is a lower semillatice with respect to $\wedge$.

We call a BCK-algebra X directed iff for every two elements x, y of X there exists an element z belonging to X such that $x \leqslant z$ and $y \leqslant z$.

Let us recall the following result obtained by T. Traczyk:

Theorem 1 (see [5]). *A commutative directed BCK-algebra is a distributive lattice with respect to $\wedge, \vee$, where $x \vee y$ is defined as $c*((c*x)\wedge(c*y))$ and c is any upper bound for x and y.*

It was shown in [4] that in a BCK-algebra the following identity holds: (1) $(x*y)*z = (x*z)*y$. Moreover, if a BCK-algebra X is directed and commutative, then (2) $(c*x)*(c*y) = y*x$, where c is an element of X such that $x \leqslant c$ and $y \leqslant c$.

Let X be a BCK-algebra. a non-empty subset I of X is called an ideal of X iff it satisfies the following conditions:

1. $0 \in I$
2. $x, y*x \in I \Rightarrow y \in I$.

It was shown in [3] that if A is a non-empty subset of an algebra X, then the set I of all elements $x \in X$ such that there exist elements $a_0, \ldots, a_n \in A$ satisfying the equation

$$(\ldots((x*a_0)*a_1)*\ldots)*a_n = 0$$

is an ideal of X generated by A. If A is a finite set, then I is called finitely generated, and if A is a one element set, the I is called principal. For a non-empty subset A of X, by $(A]$ we denote an ideal of X generated by A. If $A = \{a_1, \ldots, a_n\}$, we write $(a_1, \ldots, a_n]$ instead of $(\{a_1, \ldots, a_n\}]$.

Lemma. *(i) Let P, Q be principal ideals of a BCK-algebra X generated by a and b, respectively. Then $(P \cup Q)] = (a, b]$*

(ii) In a directed commutative BCK-algebra X $(x, y] = (x \vee y]$ for every $x, y \in X$.

Proof. We shall only show (ii), which is a non trivial part of our Lemma. The inclusion $(a_1, a_2] \subseteq (a_1 \vee a_2]$ is obvious. To prove that $(a_1 \vee a_2] \subseteq (a_1, a_2]$ it is sufficient to show that $a_1 \vee a_2 \in (a_1, a_2]$. Let us observe that it easily follows from the following:

$$((a_1 \vee a_2)*a_1)*a_2 = 0$$

Now we shall give a proof of this identity. Let $c \in X$ be an upper bound of a_1 and a_2. Then we have

$$
\begin{aligned}
((a_1 \vee a_2) * a_1) * a_2 &= ((c * ((c * a_1) * ((c * a_1) * (c * a_2)))) * a_1) * a_2 \\
\text{by } (1) &= ((c * a_1) * ((c * a_1) * ((c * a_1) * (c * a_2)))) * a_2 \\
\text{by } (1) &= ((c * a_1) * a_2) * ((c * a_1) * ((c * a_1) * (c * a_2))) \\
\text{by } (I) &\leqslant ((c * a_1) * (c * a_2)) * a_2 \\
\text{by } (2) &= (a_2 * a_1) * a_2 \\
&= 0 \text{ because } x * y \leqslant x \text{ holds in any } BCK\text{-algebra.}
\end{aligned}
$$

This completes the proof of our Lemma.

¿From the above Lemma, in a straightforward way we get the following

THEOREM 2. *Let X be a directed commutative BCK-algebra. Then an ideal I of X is finitely generated iff I is principal.*

We shall call a BCK-algebra X Noetherian iff every ideal of X is finitely generated (comp. [1]).

¿From Theorem 2 we easily obtain

THEOREM 3. *Let X be a directed commutative BCK-algebra. Then X is Noetherian iff X is a principal ideal algebra.*

The above Theorem gives a partial answer to the question posed by E. Y. Deeba in [5] whether every complete Noetherian BCK-algebra is a principal ideal algebra. Let us recall that X is a complete BCK-algebra iff every subset of X has both supremum and infimum. We can assume only directness of a BCK-algebra, which is considerably weaker than completeness.

References

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