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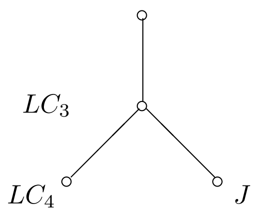
EQUIVALENTIAL FRAGMENT OF THE INFINITE VALUED LOGIC OF ŁUKASIEWICZ AND THE INTERMEDIATE LOGICS

Dedicated to
Professor Stanisław Surma
on his 50th Birthday

It is obvious that the equivalential fragment of the infinite-valued logic of Łukasiewicz ($L^{\leftrightarrow}_{\infty}$) is contained in the equivalential fragment of the classical propositional logic. From the results of Kanziński [2] it is follows that $L^{\leftrightarrow}_{\infty}$ is also contained in the equivalential fragment of the intermediate logic LC_3 determined by the pseudo-Boolean algebra of the three-element chain. In this paper we shall prove the following:

Proposition. *LC_3 is the smallest intermediate logic containing $L^{\leftrightarrow}_{\infty}$.*

In the sequel in order to simplify the notation we omit the equivalence sign $\leftrightarrow$ in formulas and we apply the convention of associating to the left, for example we write $abc(de)$ instead of $((a \leftrightarrow b) \leftrightarrow c) \leftrightarrow (d \leftrightarrow e)$. In order to denote the equivalential fragment of a logic we shall use the same symbol as for the logic itself, but with the superscript $\leftrightarrow$, for example the symbol $LC_3^{\leftrightarrow}$ denotes the equivalential fragment of the intermediate logic LC_3.



The following diagram shows the upper part of the lattice of all intermediate logics (see Smoryński [4]). The top point of the diagram represents the classical propositional logic, whose only immediate successor is the logic LC_3 of three-element chain. The Logic LC_3 has in turn two immediate successors, one of which is the logic LC_4 of four-element chain and the second is the logic J determined by the pseudo-Boolean algebra resulting from the four-element Boolean algebra by applying the Jaśkowski's Γ-operation (see Jaśkowski [1]).

It is clear that in order to prove that LC_3 is the smallest intermediate logic containing $L_\infty^\leftrightarrow$ one needs only to show that $L_\infty^\leftrightarrow \nsubseteq LC_4$ and $L_\infty^\leftrightarrow \nsubseteq J$. Recall that $L_\infty^\leftrightarrow$can be determined by the following matrix $\mathcal{L} = \langle\{0,1,2,\ldots\},\{0\},\leftrightarrow_\mathcal{L}\rangle$ where $\leftrightarrow_\mathcal{L}$ is the operation of taking the absolute value of the substraction i.e. $x \leftrightarrow_\mathcal{L} y =\mid x-y\mid$. The fragments $LC_4^\leftrightarrow$ and $J^\leftrightarrow$ can be determined by the matrices

$$\mathcal{L}_4 = \langle\{0,1,2,3\},\{0\},\leftrightarrow_\mathcal{L}\rangle \text{ and } \mathcal{F} = \langle\{0,1,2,3,4\},\{0\},\leftrightarrow_\mathcal{F}\rangle$$

with the operations $\leftrightarrow_\mathcal{L}$ and $\leftrightarrow_\mathcal{F}$ defined by the following tables:

$\leftrightarrow_\mathcal{L}$	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	2	0	3
3	3	3	3	0

$\leftrightarrow_\mathcal{L}$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	2	3	4
2	2	2	0	4	3
3	3	3	4	0	2
4	4	4	3	2	0

First we shall state an auxiliary lemma which will be very helpful in the sequel. The proof of the lemma is a routine and thus it will be omitted.

LEMMA. *Let a, b, c be arbitrary elements of the universe of the matrix $\mathcal{L}$, then the following conditions hold*

$$\text{(i)}\quad acc = \begin{cases} a & \text{if } a \leqslant c \\ 2\cdot c - a & \text{if } c < a < 2\cdot c \\ a - 2\cdot c & \text{if } 2\cdot c \leqslant a \end{cases}$$

$$\text{(ii)}\quad acca = \begin{cases} 0 & \text{if } a \leqslant c \\ 2\cdot(a-c) & \text{if } c < a < 2\cdot c \\ 2\cdot c & \text{if } 2\cdot c \leqslant a \end{cases}$$

(iii) $acc = min(a, \mid 2\cdot c - a\mid)$

(iv) $acca = min(2\cdot x, 2\cdot a \dot{-} 2\cdot c)$ – *the symbol* $(\dot{-})$ *denotes the restricted substraction,*

(v) if $a \leqslant b$ *then* $acca \leqslant bccb$.

The next step in the proof of our proposition is the following:

CLAIM 1. Put

$$\begin{aligned} A &= s(ab)(ab)s, \\ B &= A(baab)(baab)A, \\ C &= BaaB, \\ X &= CbbC. \end{aligned}$$

Then the formula X belongs to $L_\infty^{\leftrightarrow}$ and it does not belong to $LC_4^{\rightarrow}$.

PROOF. In order to verify the second part of the claim it suffices to construct a refuting valuation v of the formula X in the matrix $\mathcal{L}_4$ which can be done easily by putting $v(s) = 1$, $v(a) = 3$ and $v(b) = 2$.

In order to prove the first part let us suppose that the propositional variables of the formula X are elements of the universe of the matrix $\mathcal{L}$ such that $X > 0$. Then it follows that $B > 0$ which yields that $baab > 0$ and thus one gets that $b > a$. Since $C \leqslant 2 \cdot a$ and $0 < X \leqslant 2 \cdot (C \dot{-} b)$ then $0 < 2 \cdot (2 \cdot a \dot{-} b)$ and therefore $2 \cdot a > b$. Now we know that $2 \cdot a > b > a$ which gives that $baab = 2 \cdot (b - a)$. Since $A \leqslant 2 \cdot (b - a)$ and $0 < B \leqslant (A \dot{-} (baab)) \cdot 2$ then $0 < 2 \cdot (b - a) \dot{-} 2 \cdot (b - a)$ which is a contradiction. Q.E.D.

The final step in the proof of the proposition is the following:

CLAIM 2. Put

$$\begin{aligned} A &= s(ab)(ab)s, \\ B &= A(abb)(abb)A, \\ C &= B(baa)(baa)B, \\ D &= CaaC, \\ X &= DbbD. \end{aligned}$$

Then the formula Y belongs to $L_\infty^{\leftrightarrow}$ and it does not belong to $J^{\leftrightarrow}$.

PROOF. An easy verification of the second part proceeds by constructing a refuting valuation v of the formula Y in the matrix $\mathcal{F}$. This can be done by putting $v(s) = 1$, $v(a) = 2$ and $v(b) = 3$.

To prove the first part of the claim we assume that the propositional variables of the formula Y denote elements of the universe of the matrix $\mathcal{L}$ such that $Y > 0$. Than it follows that each one of the parts A, B, C and D must be greater then 0. Next one gets that $abb < A$, $baa < B$, $a < C$ and $b < D$. Using now the inequalities $A < 2 \cdot (ab)$, $B < 2 \cdot (abb)$, $C < 2 \cdot (baa)$, $D < 2 \cdot a$ we obtain:

(i) $abb < 2 \cdot (ab)$,
(ii) $baa < 2 \cdot (abb)$,
(iii) $a < 2 \cdot (baa)$,
(iv) $v < 2 \cdot a$.

Since $baa \leqslant b$ then by (iii) it follows:

(v) $a < 2 \cdot b$.

Since $B \leqslant 2 \cdot [A \dot{-} (abb)]$ and $A \leqslant 2 \cdot (ab)$ then $B \leqslant 2 \cdot [2 \cdot (ab) \dot{-} (abb)]$ and thus $B \leqslant 4 \cdot (ab) - 2 \cdot (abb)$ because $B > 0$. Using the inequality $baa < B$ we get:

(vi) $baa < 4 \cdot (ab) - 2 \cdot (abb)$.

Now let us consider the extra assumption $b < a$. Then $baa = b$, $abb = 2 \cdot b - a$ and $ab = a - b$. Next $b < 6 \cdot a - 8 \cdot b$ by (vi) and it follows that $3 \cdot b < 2 \cdot a$. On the other hand by (ii), $b < 4 \cdot b - 2 \cdot a$ which means that $2 \cdot a < 3 \cdot b$ – a contradiction.

Finally we have to consider the case $b \geqslant a$. Then $abb = a$, $baa = 2 \cdot a - b$ and $ab = b - a$. Applying (i) one gets $a < 2 \cdot b - 2 \cdot a$ which means that $3 \cdot a < b$. On the other hand (iii) it follows that $a < 4 \cdot a - 2 \cdot b$ and thus $2 \cdot b < 3 \cdot a$ – a contradiction. Q.E.D.

Remark. The three-variable formulas X and Y used in our proof are the best possible with respect to the number of variables involved. By means of certain results of [3] it can be shown that every equivalential formula with at most two variables which is a theorem of $L_{\infty}^{\leftrightarrow}$ must also be a theorem of the intuitionistic propositional logic.

References

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