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TOWARDS THE SOURCE OF THE NOTION OF IMPLICATION

1.

In his paper [1] professor R. Suszko defined a notion of identity connective related to the well-known notion of identity predicate. In order to compare the properties of these notions professor Suszko investigated their theories in the same propositional language.

In this paper we try to follow his pattern with the notions of ordering connective and ordering predicate.

2.

Let L be a propositional language with the binary connective $\rightarrow$ and possibly some other connectives. The symbol F_L denotes the set of all formulas of the language L built up in the usual way by means of propositional variables taken from an infinite set. If $X \subset F_L$ then $Sb(X)$ denotes the set of all substitution of formulas of X. By OR_L we denote the set of all formulas of the form $\alpha \rightarrow \beta$ where $\alpha, \beta \in F_L$.

3.

The following consequence operation C_0 in the language L defined by the inference rules (1), (2), (3) below reflects the properties of the ordering predicate:

(1) $\vdash \alpha \rightarrow \alpha$,

(2) $\alpha \rightarrow \beta, \beta \rightarrow \gamma \vdash \alpha \rightarrow \gamma$,

(3) for every $n = 1, 2, \ldots$ and for every n-ary connective φ of L, $\alpha_i \rightarrow \beta_i$, $\beta_i \rightarrow \alpha_i$ $(i = 1, \ldots, n) \vdash \varphi(\alpha_1, \ldots, \alpha_n) \rightarrow \varphi(\beta_1, \ldots, \beta_n)$.

Subsets of the set OR_L of the form $C_0(X)$ where $X \subset OR_L$ are called C_0-theories. For every $T \subset F_L$ we define a binary relation $\leqslant_T$ on F_L putting for every $\alpha, \beta \in F_L$: $\alpha \leqslant_T \beta$ iff $\alpha \to \beta \in T$.

FACT 1. If T is C_0-theory then relation $\leqslant_T$ is a quasi-ordering on F_L.

4.

Let K_L be the class of all algebras of the same type as the language L (we treat L as the absolutely free algebra of fixed type).

Let $\mathfrak{A} \in K_L$ and $\leqslant$ be an ordering on the universe of $\mathfrak{A}$. For every $h \in Hom(L, \mathfrak{A})$ we put:

$E_{\leqslant}(h) = \{\alpha \to \beta : h(\alpha) \leqslant h(\beta)\}$,
$E_{\leqslant}(\mathfrak{A}) = \bigcap\{E_{\leqslant}(h) : h \in Hom(L, \mathfrak{A})\}$,
$E(\mathcal{N}) = \bigcap\{E_{\leqslant}(\mathfrak{A}) : \leqslant \text{ is an ordering of } \mathfrak{A}\}$.

FACT 2. For every $h \in Hom(L, \mathfrak{A})$, $E_{\leqslant}(h)$ is a C_0-theory and $E(\mathfrak{A})$ is an invariant C_0-theory

$$\text{i.e. } C_0(Sb(E(\mathfrak{A}))) = E(\mathfrak{A}).$$

COMPLETENESS THEOREM. $T = C_0(Sb(X))$ *for some* $X \subset OR_L$

$$\textit{iff} \quad T = E(\mathfrak{A}) \textit{ for some } \mathfrak{A} \in K_L.$$

5.

The consequence operation C in the language L defined by the inference rules (1), (2), (3) and the rule (4) below reflects the properties of the ordering connective:

(4) $\alpha \to \beta, \alpha \vdash \beta$.

Subsets of F_L of the form $C(X)$ where $X \subset F_L$ are called C-theories.

If $\mathfrak{A} \in K_L$ and D is a subset of the universe of $\mathfrak{A}$ then the pair $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ is called a matrix.

A quasi-ordering $\leqslant$ on $\mathfrak{A}$ is called a quasi-ordering of the matrix $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ iff for every a, b of $\mathfrak{A}$, if $a \in D$ and $a \leqslant b$ then $b \in D$. In the matrix $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ we define a binary relation $\leqslant_D$ on the universe of the algebra $\mathfrak{A}$ putting $a \leqslant_D b$ iff $a \to_{\mathfrak{A}} b \in D$.

We say that the matrix $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ is a model (symbolically $\mathfrak{M} \in M_L$) if the relation $\leqslant_D$ is a quasi-ordering of the matrix $\mathfrak{M}$. A model $\mathfrak{M}$ is called normal (symbolically $\mathfrak{M} \in NM_L$) if $\leqslant_D$ is an ordering.

FACT 3. If T is C-theory then the matrix $\langle L, T \rangle$ is a model.

6.

If $h \in Hom(L, \mathfrak{A})$ nab $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ the we put:

$E(h, \mathfrak{M}) = \{\alpha : h(\alpha) \in D\}$,
$E(\mathfrak{M}) = \bigcap \{E(h, \mathfrak{M}) : h \in Hom(L, \mathfrak{A})\}$.

FACT 4. If $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ is a model then for every $h in Hom(L, \mathfrak{A})$, $E(h, \mathfrak{M})$ is a C-theory and $E(\mathfrak{M})$ is an invariant C-theory i.e. $C(Sb(E(\mathfrak{M}))) = E(\mathfrak{M})$.

COMPLETENESS THEOREM. $T = C(Sb(X))$ *for some* $X \subset F_L$

$$iff \quad T = E(\mathfrak{M}) \text{ for some } \mathfrak{M} \in NM_L.$$

References

[1] R. Suszko, *Equational logic and theories in sentential languages*, **Colloquium Mathematicum** 29 (1974), pp. 19–23.

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