Definitions in a Hyperintensional Free 2° QML

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Research Question

- What is the proof theory, and the inferential role of definitions, for a hyperintensional, free, *second-order* quantified modal logic that has the following features:
  - there are complex terms, namely definite descriptions $\iota x\varphi$ and λ-expressions $[\lambda x_1 \ldots x_n\, \varphi]$ that may fail to denote, requiring a negative free logic for the complex terms,
  - term significance/existence ($\tau\downarrow$) and identity ($\tau = \sigma$) are not primitive but rather defined,
  - all terms, including descriptions, are interpreted rigidly,
  - the axiomatization of an actuality operator $\mathcal{A}$ includes a contingent axiom ($\mathcal{A}\varphi \rightarrow \varphi$), requiring the modal logic of $\Box$ to allow for reasoning from contingencies, and
  - the theory of relations is hyperintensional and so one can't substitute necessary equivalents in all contexts?
- I answer this question by investigating the deductive system of object theory (OT), as presented in an online manuscript *Principia Logico-Metaphysica*, which has all of these features.

Why Study Object Theory?

- Systematize the domain of abstract objects axiomatically.
- Provide a mathematics-free analysis of the mathematical objects presupposed in philosophy and the sciences.
- Prove the principles about abstract objects that other theorists stipulate.
- List of Important Theorems in *Principia Logico-Metaphysica*:

  https://mally.stanford.edu/presentations/lot.pdf

The Language of 2nd-Order Modal Object Theory

- Object constants and variables: $a, b, c, \ldots \quad x, y, z, \ldots$
- Relation constants and variables:
  $P^n, Q^n, R^n, \ldots \quad F^n, G^n, H^n, \ldots$ $(n \geq 0)$
  (Use $p, q, r, \ldots$ as abbreviations when $n = 0$)
- Distinguished unary relation constant: $E!$ ‘*being concrete*’
- Basic formulas/predications:
  Where Π^n is any relation term, κ_i any individual term:
  $\Pi^n\kappa_1 \ldots \kappa_n$ (‘$\kappa_1, \ldots, \kappa_n$ exemplify Π^n’) $(n \geq 0)$
  $\kappa_1 \ldots \kappa_n\Pi^1$ (‘$\kappa_1, \ldots, \kappa_n$ encode Π^n’) $(n \geq 1)$
- Complex Formulas:
  $\neg\varphi$, $\varphi \rightarrow \psi$, $\forall\alpha\varphi$ (α any variable), $\Box\varphi$, $\mathcal{A}\varphi$
- Complex Individual and Relation Terms: (may fail to denote)
  (Rigid) Descriptions: $\iota\nu\varphi$ (ν any individual variable)
  (Rigid) λ-expressions: $[\lambda\nu_1 \ldots \nu_n\, \varphi]$ $(n \geq 0)$
- BNF: https://mally.stanford.edu/system-2nd-order-A4.pdf

The Definitions Needed for the Axioms

- All object language variables in definitions should be treated as metavariables, so that the (individual, relation) variables in the definitions can be instanced by any (individual, relation) terms. We used object language variables for readability.
- Two kinds of definitions: $\varphi \equiv_{df} \psi$ and $\tau =_{df} \sigma$
- I don't call these 'formula' vs. 'term' definitions, since formulas and terms aren't disjoint syntactic categories. (The formulas are exactly the 0-ary relations terms.)
- Though we'll be much more precise later, the idea is:
  - definitions by $\equiv_{df}$ introduce necessary equivalences
  - definitions by $=_{df}$ introduce necessary identities or nonexistent claims, depending on whether the definiens denotes.
- Existence ($\tau\!\downarrow$) and Identity ($\tau = \sigma$) will be defined by cases.

(Additional) Examples of Definitions

- Examples from system-2nd-order.pdf
- Definitions of existence and identity require 2 kinds of predication. With only classical predication:
  - you can't define $F{\downarrow} \equiv_{df} \exists xFx$, since empty properties exist.
  - you can't define $F = G$ in terms of material or necessary equivalence, since both definitions are subject to counterexamples.
- Defining constants via an empty definiens is allowed:
  - $e =_{df} \iota x(Fx \ \& \ \neg Fx)$
  - $K =_{df} [\lambda x\, \exists F(xF \ \& \ \neg Fx)]$
- Given the definition of $\downarrow$, the system will imply $\neg e{\downarrow}$ and $\neg K{\downarrow}$.
- Then, the free logic will prevent us from instantiating e in $\forall x\varphi$ to obtain φ^{e}_{x}, or instantiate K in $\forall F\varphi$ to obtain φ^{K}_{F}.
- These facts will become clearer once we see the axioms, rules, metarules (including metarules governing the inferential roles of $\equiv_{df}$ and $=_{df}$) and some important theorems.

Distinguished Axioms: Predicate Logic

- τ ranges over terms, α, β range over variables, and when φ^{τ}_{α}, both τ and α have the same type.
- Logic of Quantification has 3 distinctive axioms for negative free logic (see system-2nd-order.pdf):
- The first is standard:
  - $\forall\alpha\varphi \rightarrow (\tau\!\downarrow \rightarrow \varphi^{\tau}_{\alpha})$, provided …

  Recall: $\tau\!\downarrow$ is defined! In some systems, $\downarrow$ is primitive (Feferman 1995); in others, $\tau\!\downarrow$ is replaced by $\exists\beta(\beta\!=\!\tau)$, where $=$ is primitive.
- The second:
  - $\tau\!\downarrow$, provided …

  So $\iota x\varphi\!\downarrow$ is not asserted, and $[\lambda x_1 \ldots x_n\, \varphi]\!\downarrow$ is not asserted when an x_i is in encoding position in φ, e.g., $[\lambda x\, \exists F(xF \,\&\, \neg Fx)]$.
- Convention: Free variables in encoding position in the definiens are considered to be in encoding position in definiendum.
- The third: predications imply the existence of the primary terms.

Distinguished But Familiar Axioms

- Substitution of identicals is unrestricted (no term changes its denotation in modal contexts).
- The necessitations of the axiom $\mathcal{A}\varphi \rightarrow \varphi$ are not asserted; this is a logical truth that isn't necessary. (Zalta 1988)
- To guarantee descriptions are rigid, $\mathcal{A}$ is used in the axiom:
  - $y = \iota x\varphi \equiv \forall x(\mathcal{A}\varphi \equiv x = y)$

  This is a minor variant of Hintikka (1959, 83) sentence (7b), which Lambert (1962, 53) adopts in sentence (6).
- If τ is a description of the form $\iota x\varphi$, then

$$\boldsymbol{d}_{\mathcal{I},f}(\tau) = \begin{cases} \boldsymbol{o}, \text{ if } \boldsymbol{w}_\alpha \models_{\mathcal{I},f[x/\boldsymbol{o}]} \varphi \;\&\; \forall \boldsymbol{o}'(\boldsymbol{w}_\alpha \models_{\mathcal{I},f[x/\boldsymbol{o}']} \varphi \rightarrow \boldsymbol{o}' = \boldsymbol{o}) \\ \text{undefined, otherwise} \end{cases}$$

where $\boldsymbol{o}, \boldsymbol{o}'$ are entities in the domain of individuals **D**.

Distinguished But Familiar Axioms: Relations

- Semantically, $[\lambda x_1 \ldots x_n\, \varphi]$ denotes a primitive relation $\boldsymbol{r}^n$ whose exemplification extension at a world $\boldsymbol{w}$ consists of all and only those n-tuples that satisfy φ at $\boldsymbol{w}$, i.e., where τ is $[\lambda x_1 \ldots x_n\, \varphi]$, then

$$\boldsymbol{d}_{\mathcal{I},f}(\tau) = \begin{cases} \bar{\varepsilon}\boldsymbol{r}^n \forall \boldsymbol{w} \forall \boldsymbol{o}_1 \ldots \forall \boldsymbol{o}_n (\langle \boldsymbol{o}_1, \ldots, \boldsymbol{o}_n \rangle \in \mathbf{ext}_{\boldsymbol{w}}(\boldsymbol{r}) \equiv \boldsymbol{w} \models_{\mathcal{I},f[x_i/\boldsymbol{o}_i]_{i=1}^n} \varphi), \\ \qquad \text{if there is one} \\ \text{undefined, otherwise} \end{cases}$$

  where $\bar{\varepsilon}\boldsymbol{r}\Phi$ is a semantic epsilon term that denotes a relation $\boldsymbol{r}$.
- α- and β-Conversion are conditionalized since λ-expressions may fail to denote.
- η-Conversion applies to *elementary* λ-expressions.
- The final axiom for relations intuitively asserts: if $[\lambda x_1 \ldots x_n\, \varphi]$ denotes, and ψ (which might have x_i in encoding position) is necessarily and universally equivalent to φ, then $[\lambda x_1 \ldots x_n\, \psi]$ denotes.

The Axioms of Encoding

- 4 axioms (see system-2nd-order.pdf):
  - n-ary encoding predication is equivalent to a conjunction of n *unary* encoding predications.
  - Ordinary objects don't encode properties.
  - Encoding is rigid.
  - Comprehension for abstract objects.
- Once the deductive system is in place, comprehension and identity for abstract objects implies:
  - $\vdash \exists!x(A!x \,\&\, \forall F(xF \equiv \varphi))$, provided x isn't free in φ
  - $\vdash \iota x(A!x \,\&\, \forall F(xF \equiv \varphi))\!\downarrow$, provided x isn't free in φ

  These descriptions are canonical!
- Examples of abstract objects defined by canonical descriptions:
  - $\boldsymbol{c}_y \;=_{df}\; \iota x(A!x \,\&\, \forall F(xF \equiv Fy))$
  - $\Phi_G \;=_{df}\; \iota x(A!x \,\&\, \forall F(xF \equiv \Box\forall y(Gy \rightarrow Fy)))$
  - $\epsilon G \;=_{df}\; \iota x(A!x \,\&\, \forall F(xF \equiv \forall y(Gy \equiv Fy)))$
  - $\#G \;=_{df}\; \iota x(A!x \,\&\, \forall F(xF \equiv F \approx_D G))$
  - $y \oplus z =_{df}\; \iota x(A!x \,\&\, \forall F(xF \equiv yF \vee zF))$

Primitive Rules, Metarules, and Derived Rules

- Primitive Rule of Inference: Modus Ponens
  - $\varphi, \varphi \rightarrow \psi \,/\, \psi$
- Two Kinds of Derivations and Proofs:
  - Standard ($\Gamma \vdash \varphi$ and $\vdash \varphi$): any sequence ... (defined as usual)
  - Strict ($\Gamma \vdash_{\Box} \varphi$ and $\vdash_{\Box} \varphi$): no dependence on the contingent schema $\mathcal{A}\varphi \rightarrow \varphi$

  For simplicity: we use $\vdash$ unless we specifically need $\vdash_{\Box}$, and we affix a $\star$ to the number of any theorem (or derivation) that depends on $\mathcal{A}\varphi \rightarrow \varphi$.
- Metatheorem: If $\Gamma \vdash_{\Box} \varphi$, then $\Gamma \vdash \varphi$.
- Derived Metarules:
  - GEN: If $\Gamma \vdash \varphi$ and α isn't free in Γ, then $\Gamma \vdash \forall\alpha\varphi$.
  - RN: If $\Gamma \vdash_{\Box} \varphi$, then $\Box\Gamma \vdash_{\Box} \Box\varphi$.
  - RA: If $\Gamma \vdash \varphi$, then $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$.

Inferential Role of Definitions by Equivalence

- Primitive Rule for $\equiv_{df}$. The idea is that these definitions introduce the (universal, modal, actualized) closures of equivalences:
  - **Rule of Definition by Equivalence**: A definition-by-$\equiv$ of the form $\varphi \equiv_{df} \psi$ introduces the closures of $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$ as necessary axioms.
- Examples:
  - $\mathit{ContingentlyExemplifies}(x, F) \;\; \equiv_{df} \;\; Fx \;\&\; \neg\Box Fx$
  - $\mathit{CondNecFor}(F, x) \;\; \equiv_{df} \;\; F\!\downarrow \;\&\; x\!\downarrow \;\&\; (Fx \rightarrow \Box Fx)$

  The additions of $\downarrow$ clauses ensures that the definiendum is false when the definition is instanced by an empty term.

The Problem for Defining Individual Constants in QML

- Suppes 1957 (159–60), Gupta 2023 (§2.4), for non-modal logics:
  - $\delta = x \equiv_{df} \varphi$, provided $\vdash \exists!x\varphi$
- Problem: if such definitions introduce necessary equivalences, then $\vdash \Box\exists x\varphi$ (proof in Appendix); you can't assert $\Diamond\neg\exists x\varphi$.
- Suggestion 1: require $\vdash \Box\exists!x\varphi$. But this fails: the constant δ might denote something different in a different modal contexts.
- Suggestions 2: require $\vdash \exists!x\Box\varphi$. This also fails: $\exists!x\Box\varphi$ can be true while $\exists!x\varphi$ is not. Consider two objects a, b and two worlds $\boldsymbol{w}_\alpha, \boldsymbol{w}_1$, where Pa at $\boldsymbol{w}_\alpha$ and $\boldsymbol{w}_1$, but Pb only at $\boldsymbol{w}_\alpha$. Then $\exists!x\Box Pa$, but $\neg\exists!xPx$.
- You need $\vdash \exists!x\Box\varphi$ and $\vdash \exists!x\varphi$ to introduce $\delta = x \equiv_{df} \varphi$ in QML.
- But things are simpler in OT: $\delta =_{df} \iota x\varphi$. You introduce a rigid constant in terms of a rigid description, where the inferential role is stipulated as follows:

Inferential Role of Definitions by Identity

- Primitive Rule for $=_{df}$. The idea is that $\tau =_{df} \sigma$ introduces (the closures of) conjunctions of conditionals as necessary axioms:
  - $(\sigma\downarrow \rightarrow \tau = \sigma) \; \& \; (\neg\sigma\downarrow \rightarrow \neg\tau\downarrow)$
- **Rule of Definition by Identity** (Two-Free Variables)
  Whenever τ_1 and τ_2 are any terms substitutable, respectively, for α_1 and α_2 in $\sigma(\alpha_1, \alpha_2)$, then a definition of the form $\tau(\alpha_1, \alpha_2) =_{df} \sigma(\alpha_1, \alpha_2)$ introduces (the closures of) the following axiom schema:
  $$(\sigma(\tau_1, \tau_2)\downarrow \rightarrow \tau(\tau_1, \tau_2) = \sigma(\tau_1, \tau_2)) \; \& \; (\neg\sigma(\tau_1, \tau_2)\downarrow \rightarrow \neg\tau(\tau_1, \tau_2)\downarrow)$$
- Example (assume real number theory for this example):
  - $x/y =_{df} \imath z(x = y \cdot z)$ introduces the (closures of the) axiom:
    - $(\imath z(x = y \cdot z)\downarrow \rightarrow x/y = \imath z(x = y \cdot z)) \; \& \; (\neg\imath z(x = y \cdot z)\downarrow \rightarrow \neg(x/y)\downarrow)$
  - When $y = 0$, the definition of x/y also introduces:
    - $(\imath z(x = 0 \cdot z)\downarrow \rightarrow x/0 = \imath z(x = 0 \cdot z)) \; \& \; (\neg\imath z(x = 0 \cdot z)\downarrow \rightarrow \neg(x/0)\downarrow)$
  - It is provable in $\mathbb{R}$ that $\neg\exists! z(x = 0 \cdot z)$, even when x is 0. So given $\vdash \neg\imath z(x = 0 \cdot z)\downarrow$, it follows that $\neg(x/0)\downarrow$ will be derivable.

Some Key Theorems: Existence and Necessary Existence

- $\vdash \varphi\downarrow$, where φ is any formula.
  Proof Sketch: By definition, $p\downarrow \equiv [\lambda x\, p]\downarrow$. Since variables function as metavariables in definitions, we have $\Pi^0\downarrow \equiv [\lambda \nu\, \Pi^0]\downarrow$, provided ν isn't free in Π^0. But since ν isn't free in Π^0, the variable bound by the λ isn't in encoding position in the matrix, and so the logic asserts $[\lambda \nu\, \Pi^0]\downarrow$. Hence, $\Pi^0\downarrow$. But all and only formulas φ are 0-ary relation terms, by the BNF. So $\varphi\downarrow$. $\bowtie$
- $\vdash \tau\downarrow \rightarrow \Box\tau\downarrow$, where τ is any term
  Proof Sketch: The modal closures of $\alpha\downarrow$ are axioms, where α is any variable. So $\Box\alpha\downarrow$ is an axiom. By GEN, $\forall\alpha(\Box\alpha\downarrow)$. Now let φ be $\Box\alpha\downarrow$. Then:

  $$\forall\alpha(\Box\alpha\downarrow) \rightarrow (\tau\downarrow \rightarrow \Box\tau\downarrow),\text{ where } \tau \text{ is substitutable for } \alpha \text{ in } \Box\alpha\downarrow$$

  But every term τ of the same type as α is substitutable for α in $\Box\alpha\downarrow$. So $\tau\downarrow \rightarrow \Box\tau\downarrow$. $\bowtie$
- This does *not* imply that ordinary objects are necessarily concrete or imply that relations can't be contingently exemplified.

Key Theorems: Identity and Necessity of Identity

- $\vdash$ Identity is an equivalence condition:

  $$\alpha = \alpha$$
  $$\alpha = \beta \rightarrow \beta = \alpha$$
  $$(\alpha = \beta \mathbin{\&} \beta = \gamma) \rightarrow \alpha = \gamma$$

  Proof Sketch on tablet.

- $\vdash \alpha = \beta \rightarrow \Box \alpha = \beta$, when α, β are variables of the same type.
  Proof: Follow Kripke's proof, but note: (a) = is not primitive, and (b) our theorem holds for relations as well.

- $\vdash \tau = \sigma \rightarrow (\tau\downarrow \mathbin{\&} \sigma\downarrow)$
  Proof: All of the definitientia in the definition of = by cases invoke predications (which require that the terms denote) or existence clauses.

- $\vdash \tau = \sigma \rightarrow \Box \tau = \sigma$, where τ and σ are any terms of the same type.

Key Theorems: Existence and Identity Related

- $\vdash \tau\!\downarrow \equiv \exists\beta(\beta = \tau)$, provided β doesn't occur free in τ
- Proof Sketch: Consider any term τ in which β doesn't occur free. ($\rightarrow$) Assume $\tau\!\downarrow$. Since the definitions of identity (for individuals and relations) imply $\alpha = \alpha$, it is a theorem that $\forall\alpha(\alpha = \alpha)$. Instantiating to τ yields $\tau = \tau$. Since β doesn't occur free in τ, it follows that $\exists\beta(\beta = \tau)$, by $\exists$I.

  ($\leftarrow$) Assume $\exists\beta(\beta = \tau)$. Let the simple constant σ denote an arbitrary such entity, so that we know $\sigma = \tau$. But the definitions of identity (for individuals and relations) then imply both $\sigma\!\downarrow$ and $\tau\!\downarrow$. The latter is sufficient. $\bowtie$
- Though this seems like a trivial theorem, its interest lies in the facts that (a) $\downarrow$ and $=$ are both defined in terms of predication and quantification in a modal context, for both individuals and relations, and (b) existence and identity are correctly related.

Additional Key Theorems

- $\star \vdash y = \iota x\varphi \equiv \varphi^y_x \ \&\ \forall x(\varphi \to x = y)$, provided y is substitutable for x and not free in φ (Hintikka 1959)

  Proof Sketch: The axiom for descriptions is

  (ϑ) $y = \iota x\varphi \ \equiv\ \forall x(\mathcal{A}\varphi \equiv x = y)$

  But, $\forall x(\mathcal{A}\varphi \equiv x = y) \equiv \forall x(\varphi \equiv x = y)$ is a $\star$-theorem, given that $\mathcal{A}\varphi \equiv \varphi$ is a $\star$-theorem. So $y = \iota x\varphi \ \equiv\ \forall x(\varphi \equiv x = y)$. Since y is substitutable for x and not free in φ we know $\forall x(\varphi \equiv x = y)$ is (necessarily) equivalent to $\varphi^y_x \ \&\ \forall x(\varphi \to x = y)$. So $y = \iota x\varphi \equiv \varphi^y_x \ \&\ \forall x(\varphi \to x = y)$. $\bowtie$

- $\star \vdash \iota x\varphi\downarrow \equiv \exists! x\varphi$

  Proof Sketch: ($\to$) Assume $\iota x\varphi\downarrow$. Then $\exists y(y = \iota x\varphi)$, for y not free in φ. Suppose $a = \iota x\varphi$. By Hintikka's schema, $a = \iota x\varphi \equiv \varphi^a_x \ \&\ \forall x(\varphi \to x = a)$. So $\varphi^a_x \ \&\ \forall x(\varphi \to x = a)$. Pick z substitutable for x and not free in φ, for alphabetic variant: $\varphi^a_x \ \&\ \forall z(\varphi^z_x \to z = a)$. Hence, $\exists! x\varphi$. ($\leftarrow$) By analogous reasoning. $\bowtie$

Russell's Axiom is a ⋆-Theorem

- $\star \vdash \psi^{\imath x\varphi}_{x} \equiv \exists x(\varphi \mathbin{\&} \forall z(\varphi^{z}_{x} \rightarrow z = x) \mathbin{\&} \psi)$, provided …
  Proof Sketch: (→) Assume $\psi^{\imath x\varphi}_{x}$. Since ψ is an exemplification or encoding formula, it follows that $\imath x\varphi\downarrow$. Hence $\exists z(z = \imath x\varphi)$. Suppose $a = \imath x\varphi$. But we know the following, by applying GEN to Hintikka's ⋆-schema:
  $$\forall z(z = \imath x\varphi \equiv (\varphi^{z}_{x} \mathbin{\&} \forall x(\varphi \rightarrow x = z)))$$
  So $a = \imath x\varphi \equiv \varphi^{a}_{x} \mathbin{\&} \forall x(\varphi \rightarrow x = a)$. Hence, $\varphi^{a}_{x} \mathbin{\&} \forall x(\varphi \rightarrow x = a)$, which yields the alphabetic variant: $\varphi^{a}_{x} \mathbin{\&} \forall z(\varphi^{z}_{x} \rightarrow z = a)$. Since both $a = \imath x\varphi$ and $\psi^{\imath x\varphi}_{x}$, it follows that ψ^{a}_{x}. Hence:
  $$\varphi^{a}_{x} \mathbin{\&} \forall z(\varphi^{z}_{x} \rightarrow z = a) \mathbin{\&} \psi^{a}_{x}$$
  So $\exists x(\varphi \mathbin{\&} \forall z(\varphi^{z}_{x} \rightarrow z = x) \mathbin{\&} \psi)$.
  (←) Assume $\exists x(\varphi \mathbin{\&} \forall z(\varphi^{z}_{x} \rightarrow z = x) \mathbin{\&} \psi)$. Suppose:
  $$\varphi^{b}_{x} \mathbin{\&} \forall z(\varphi^{z}_{x} \rightarrow z = b) \mathbin{\&} \psi^{b}_{x}$$
  But independently, by Hintikka's ⋆-schema:
  $b = \imath x\varphi \equiv \varphi^{b}_{x} \mathbin{\&} \forall x(\varphi \rightarrow x = b)$. By alphabetic variant:
  $b = \imath x\varphi \equiv \varphi^{b}_{x} \mathbin{\&} \forall z(\varphi^{z}_{x} \rightarrow z = b)$. So $b = \imath x\varphi$. Then, $\psi^{\imath x\varphi}_{x}$. ⋈

Proof of Lambert's Axiom

- $\vdash y = \iota x(x = y)$ (Lambert 1962, Axiom 2)

  Proof Sketch: Show (A) $\iota x(x = y)\downarrow$, and then (B) $y = \iota x(x = y)$.

  (A) We have to show $\exists F(F\iota x(x = y))$. Let $L =_{df} [\lambda x\, E!x \rightarrow E!x]$.

  Show: $L\iota x(x = y)$. So by *modally strict* Russell, show:

  $$\exists x(\mathcal{A}x = y \,\&\, \forall z(\mathcal{A}z = y \rightarrow z = x) \,\&\, Lx)$$

  Since $y\downarrow$, let y be our witness; so it suffices to show:

  (ϑ) $\mathcal{A}y = y \,\&\, \forall z(\mathcal{A}z = y \rightarrow z = y) \,\&\, Ly$

  Given the necessity of identity, we know:

  (ξ) $y = y \equiv \mathcal{A}y = y$

  (ζ) $z = y \equiv \mathcal{A}z = y$

  The 1st conjunct of (ϑ) follows from (ξ) and $y = y$. The 2nd conjunct of (ϑ) follows, by GEN, from the right-to-left direction of (ζ). The third conjunct of (ϑ) follows from the fact that $\forall zLz$.

  (B) We know $x = y \equiv \mathcal{A}x = y$, i.e., $\mathcal{A}x = y \equiv x = y$. By GEN, $\forall x(\mathcal{A}x = y \equiv x = y)$. Then by the OT *axiom* for descriptions: $y = \iota x(x = y)$. ⋈

Observations

- These classical principles for existence (↓) and identity (=) hold for both both 1st- and 2nd-order terms.
- The principles $\tau\downarrow \rightarrow \Box\tau\downarrow$ and $\tau = \sigma \rightarrow \Box\tau = \sigma$ are consistent with the contingency of ordinary objects.
- The free logic applies only to complex terms (simple terms behave classically) and a key principle of free logic has been derived generally: $\tau\downarrow \equiv \exists\beta(\beta = \tau)$.
- The classical principles governing descriptions have been derived, but in terms of ↓ and = *defined* in terms of predication and quantification.
- Relations preserved throughout as hyperintensional; definitions by ≡ preserve hyperintensionality.
- There are no "conditional definitions"; you don't have to prove that some term exists before you can use it in a definition by =. The inferential role of definitions by = ensures that definitions are conservative and eliminable.

The Modal Problem for $\delta = x \equiv_{df} \varphi$

- If $\equiv_{df}$ introduces necessitatable equivalences, the above definition implies $\Box\exists x\varphi$ given $\exists!x\varphi$ (you can't assert $\Diamond\neg\exists x\varphi$).
- Suppose $\delta = x \equiv_{df} \varphi$ and $\Diamond\neg\exists x\varphi$. By hypothesis, $\vdash \exists!x\varphi$, so suppose it is b. Then, φ^b_x and $\forall y(\varphi^y_x \rightarrow y = b)$. Then by definition:

  $(\vartheta)\quad \delta = b \equiv \varphi^b_x$

  and by Rule RN :

  $(\xi)\quad \Box(\delta = b \equiv \varphi^b_x)$

  Clearly, $\delta\downarrow$, since $\exists!x\varphi$. Then by the necessity of identity,

  $(\zeta)\quad \delta = b \rightarrow \Box\delta = b$

  So $\varphi^b_x \rightarrow \Box\varphi^b_x$, by the following chain: $\varphi^b_x \rightarrow \delta = b$, by (ϑ); $\delta = b \rightarrow \Box\delta = b$, by (ζ); and $\Box\delta = b \rightarrow \Box\varphi^b_x$, by (ξ) and the modal theorem $\Box(\psi \equiv \chi) \rightarrow (\Box\psi \equiv \Box\chi)$. Having thus established that $\varphi^b_x \rightarrow \Box\varphi^b_x$, then our assumption φ^b_x implies $\Box\varphi^b_x$. Hence, by $\exists$I, $\exists x\Box\varphi$. So by the Buridan formula, $\Box\exists x\varphi$, which contradicts $\Diamond\neg\exists x\varphi$. $\bowtie$

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