Towards a general proof theory of term-forming operators 2

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2. An approach developed by Neil Tennant.

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EXT: $\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \tau x\varphi(x) = \tau x\psi(x)$
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The formalisation GT1: to GC add:

$$(Ext)\ \frac{\varphi(a), \Gamma \Rightarrow \Delta, \psi(a) \qquad \psi(a), \Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \tau x\varphi(x) = \tau x\psi(x)}$$

$$(AV)\ \frac{\tau x\varphi(x) = \tau y\varphi(y), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

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If we add existence predicate E, which is usually defined as $Et := \exists x(x = t)$, then the following hold for FFOLI:

$\forall x\varphi \wedge Et \rightarrow \varphi[x/t]$

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Moreover, in NFFOLI additionally atomic formulae with nondenoting terms are false which implies that:

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and also:

$\varphi(t) \rightarrow Et$
for any atomic formula φ.

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Based on the following ND rules:

τI If $\varphi(a), Ea \vdash aRt$ and $aRt \vdash \varphi(a)$ and Et, then $t = \tau x\varphi(x)$;
$\tau E1$ If $t = \tau x\varphi(x)$ and $\varphi(b)$ and Eb, then bRt
$\tau E2$ If $t = \tau x\varphi(x)$, then Et
$\tau E3$ If $t = \tau x\varphi(x)$ and bRt, then $\varphi(b)$

where a is an eigenvariable, and R is the specific relation involved in the characterisation of τ; e.g. $=$ for ι, $\in$ for set builder.

Tennant's approach:

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The corresponding sequent rules:

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$$(\Rightarrow \tau)\ \frac{\Gamma \Rightarrow \Delta, Et \qquad Ea, \varphi(a), \Gamma \Rightarrow \Delta, aRt \qquad aRt, \Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, t = \tau x \varphi(x)}$$

where a is not in Γ, Δ, φ

$$(\Rightarrow \tau E1)\ \frac{\Gamma \Rightarrow \Delta, Eb \qquad \Gamma \Rightarrow \Delta, \varphi(b) \qquad \Gamma \Rightarrow \Delta, t = \tau x \varphi(x)}{\Gamma \Rightarrow \Delta, bRt}$$

$$(\Rightarrow \tau E2)\ \frac{\Gamma \Rightarrow \Delta, t = \tau x \varphi(x)}{\Gamma \Rightarrow \Delta, Et}$$

$$(\Rightarrow \tau E3)\ \frac{\Gamma \Rightarrow \Delta, bRt \qquad \Gamma \Rightarrow \Delta, t = \tau x \varphi(x)}{\Gamma \Rightarrow \Delta, \varphi(b)}$$

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More standard sequent rules:

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To get more standard SC we apply Rule-maker lemma and obtain left introduction rules for τ:

$$(\tau \Rightarrow 1)\ \frac{\Gamma \Rightarrow \Delta, Eb \qquad \Gamma \Rightarrow \Delta, \varphi(b) \qquad bRt, \Gamma \Rightarrow \Delta}{t = \tau x \varphi(x), \Gamma \Rightarrow \Delta}$$

$$(\tau \Rightarrow 2)\ \frac{Et, \Gamma \Rightarrow \Delta}{t = \tau x \varphi(x), \Gamma \Rightarrow \Delta}$$

$$(\tau \Rightarrow 3)\ \frac{\Gamma \Rightarrow \Delta, bRt \qquad \varphi(b), \Gamma \Rightarrow \Delta}{t = \tau x \varphi(x), \Gamma \Rightarrow \Delta}$$

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As a result we obtain the following rules:

$$(\Rightarrow \tau)\ \frac{\varphi(a), \Gamma \Rightarrow \Delta, aRt \qquad aRt, \Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, t = \tau x \varphi(x)}$$

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$$(\tau \Rightarrow 1)\ \frac{\Gamma \Rightarrow \Delta, \varphi(b) \qquad bRt, \Gamma \Rightarrow \Delta}{t = \tau x \varphi(x), \Gamma \Rightarrow \Delta}$$

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In general what we obtain with these rules is equivalent to the following principle:

$$\forall y(y = \tau x\varphi(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow xRy)$$

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On the ground of CFOLI it is equivalent to:

$t = \tau x\varphi(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow xRt)$.

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for which we demonstrate syntactically the equivalence with the stated rules.

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In one direction we have:

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(\Rightarrow \forall)\ \dfrac{(\Rightarrow\leftrightarrow)\ \dfrac{(\tau\Rightarrow)\ \dfrac{\varphi[x/a] \Rightarrow \varphi[x/a] \qquad aRt \Rightarrow aRt}{t = \tau x\varphi(x), \varphi[x/a] \Rightarrow aRt} \qquad \dfrac{aRt \Rightarrow aRt \qquad \varphi[x/a] \Rightarrow \varphi[x/a]}{t = \tau x\varphi(x), aRt \Rightarrow \varphi[x/a]}}{t = \tau x\varphi(x) \Rightarrow \varphi[x/a] \leftrightarrow aRt}}{t = \tau x\varphi(x) \Rightarrow \forall x(\varphi(x) \leftrightarrow xRt)}
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In the second direction:

$$
(\Rightarrow\tau)\ \dfrac{(\forall\Rightarrow)\ \dfrac{(\leftrightarrow\Rightarrow)\ \dfrac{aRt\Rightarrow aRt \qquad \varphi[x/a]\Rightarrow\varphi[x/a]}{\varphi[x/a]\leftrightarrow aRt, aRt\Rightarrow\varphi[x/a]}}{\forall x(\varphi(x)\leftrightarrow xRt), aRt\Rightarrow\varphi[x/a]} \qquad \dfrac{\dfrac{\varphi[x/a]\Rightarrow\varphi[x/a] \qquad aRt\Rightarrow aRt}{\varphi[x/a]\leftrightarrow aRt,\varphi[x/a]\Rightarrow aRt}}{\forall x(\varphi(x)\leftrightarrow xRt),\varphi[x/a]\Rightarrow aRt}}{\forall x(\varphi(x)\leftrightarrow xRt)\Rightarrow t=\tau x\varphi(x)}
$$

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Derivability of the specific rules is straightforward. Notice that from the principle as an additional axiom we obtain:

(a) $t = \tau x\varphi(x) \Rightarrow \forall x(\varphi(x) \leftrightarrow xRt)$ and

(b) $\forall x(\varphi(x) \leftrightarrow xRt) \Rightarrow t = \tau x\varphi(x)$.

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From the premisses of any variant of $(\tau \Rightarrow)$ by W we deduce:

$$(\leftrightarrow\Rightarrow)\ \dfrac{\Gamma \Rightarrow \Delta, bRt, \varphi[x/b] \qquad bRt, \varphi[x/b], \Gamma \Rightarrow \Delta}{(\forall \Rightarrow)\ \dfrac{\varphi[x/b] \leftrightarrow bRt, \Gamma \Rightarrow \Delta}{\forall x(\varphi(x) \leftrightarrow xRt), \Gamma \Rightarrow \Delta}}$$

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which, by cut with (a) yields the conclusion of $(\tau \Rightarrow)$. In a similar way we deduce $\Gamma \Rightarrow \Delta, \forall x(\varphi(x) \leftrightarrow xRt)$ from premisses of $(\Rightarrow \tau)$, and by cut with (b) we obtain the conclusion of this rule.

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$$
\dfrac{\dfrac{\dfrac{\dfrac{aR\tau x\varphi(x) \Rightarrow aR\tau x\varphi(x) \qquad \varphi[x/a], \varphi[x/a] \leftrightarrow \psi[x/a] \Rightarrow \psi[x/a]}{\tau x\varphi(x) = \tau x\varphi(x), \varphi[x/a] \leftrightarrow \psi[x/a], aR\tau x\varphi(x) \Rightarrow \psi[x/a]}(\tau\Rightarrow)}{\varphi[x/a] \leftrightarrow \psi[x/a], aR\tau x\varphi(x) \Rightarrow \psi[x/a]}(=\Rightarrow)}{\forall x(\varphi(x) \leftrightarrow \psi(x)), aR\tau x\varphi(x) \Rightarrow \psi[x/a]}(\forall\Rightarrow) \qquad D}{\forall x(\varphi(x) \leftrightarrow \psi(x)) \Rightarrow \tau x\varphi(x) = \tau x\psi(x)}(\Rightarrow\tau)
$$

where the second leaf is directly provable and D is an analogous proof of $\forall x(\varphi(x) \leftrightarrow \psi(x)), \psi[x/a] \Rightarrow aR\tau x\varphi(x)$.

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$$
\dfrac{(=\Rightarrow)\ \dfrac{(\tau\Rightarrow)\ \dfrac{aR\tau x\varphi(x) \Rightarrow aR\tau x\varphi(x) \qquad \varphi[x/a] \Rightarrow \varphi[y/a]}{\tau x\varphi(x) = \tau x\varphi(x), aR\tau x\varphi(x) \Rightarrow \varphi[y/a]}}{aR\tau x\varphi(x) \Rightarrow \varphi[y/a]} \qquad \dfrac{\dfrac{\varphi[y/a] \Rightarrow \varphi[x/a] \qquad aR\tau x\varphi(x) \Rightarrow aR\tau x\varphi(x)}{\tau x\varphi(x) = \tau x\varphi(x), \varphi[y/a] \Rightarrow aR\tau x\varphi(x)}}{\varphi[y/a] \Rightarrow aR\tau x\varphi(x)}}{\Rightarrow \tau x\varphi(x) = \tau y\varphi(y)}\ (\Rightarrow\tau)
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Note that $\varphi[x/a]$ and $\varphi[y/a]$ are identical.

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One may even prove the converse or EXT:

$$(\Rightarrow\forall)\;\dfrac{\dfrac{(\tau\Rightarrow)\;\dfrac{(=\Rightarrow)\;\dfrac{(\tau\Rightarrow)\;\dfrac{\varphi[x/a]\Rightarrow\varphi[x/a] \qquad aR\tau x\varphi(x)\Rightarrow aR\tau x\varphi(x)}{\tau x\varphi(x)=\tau x\varphi(x),\varphi[x/a]\Rightarrow aR\tau x\varphi(x)}}{\varphi[x/a]\Rightarrow aR\tau x\varphi(x)} \qquad \psi[x/a]\Rightarrow\psi[x/a]}{\tau x\varphi(x)=\tau x\psi(x),\varphi[x/a]\Rightarrow\psi[x/a]} \qquad D}{\tau x\varphi(x)=\tau x\psi(x)\Rightarrow\varphi[x/a]\leftrightarrow\psi[x/a]}}{\tau x\varphi(x)=\tau x\psi(x)\Rightarrow\forall x(\varphi(x)\leftrightarrow\psi(x))}$$

where D is a similar proof of $\tau x\varphi(x)=\tau x\psi(x),\psi[x/a]\Rightarrow\varphi[x/a]$.

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To realize how strong is this principle on the ground of CFOLI notice that when t is instantiated with $\tau x\varphi(x)$ we obtain:

$$\tau x\varphi(x) = \tau x\varphi(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow xR\tau x\varphi(x)).$$

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For several term-forming operators, at least on the ground of CFOLI, it is too strong.

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3. $\forall x(Ax \wedge \neg Ax \leftrightarrow x = \imath x(Ax \wedge \neg Ax))$ 1, 2
4. $A(\imath x(Ax \wedge \neg Ax)) \wedge \neg A(\imath x(Ax \wedge \neg Ax)) \leftrightarrow \imath x(Ax \wedge \neg Ax) = \imath x(Ax \wedge \neg Ax))$ 3
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However, even on the basis of CFOLI one may introduce several restrictions which can prevent us against troubles. We will illustrate this with abstract operator.

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Language with $\in$ primitive.

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Two axioms:

Abs $\forall x(x \in \{y : \varphi(y)\} \leftrightarrow \varphi(y/x))$, φ stratified.
Ext $\forall xy(x = y \rightarrow (\varphi(x) \leftrightarrow \varphi(y)))$

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ExtAx $\forall xy(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$

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Tennant is using $=$ primitive and works with NFFOLI.
This means that if we use Tennant's-style rules in the context of CFOLI we need simplified rules for set builders (for GCFOLI):

$$(\Rightarrow :)\ \frac{\varphi(a), \Gamma \Rightarrow \Delta, a \in t \qquad a \in t, \Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, t = \{x : \varphi(x)\}}$$

where a is not in Γ, Δ, φ and φ is stratified.

$$(: \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi(b) \qquad b \in t, \Gamma \Rightarrow \Delta}{t = \{x : \varphi(x)\}, \Gamma \Rightarrow \Delta}$$

$$(: \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, b \in t \qquad \varphi(b), \Gamma \Rightarrow \Delta}{t = \{x : \varphi(x)\}, \Gamma \Rightarrow \Delta}$$

where t is any term and φ is stratified.

Application to set-builders

Application to set-builders

If you add these rules to GCFOLI (1 approach to identity) you obtain $(ExtAx)$ for free – it is provable:

$$
(\Rightarrow\forall)\dfrac{(\Rightarrow\to)\dfrac{(LL)\dfrac{(\Rightarrow:)\dfrac{(\forall\Rightarrow)\dfrac{a\in c\leftrightarrow a\in d, a\in c\Rightarrow a\in d}{\forall z(z\in c\leftrightarrow z\in d), a\in c\Rightarrow a\in d}\qquad \dfrac{a\in c\leftrightarrow a\in d, a\in d\Rightarrow a\in c}{\forall z(z\in c\leftrightarrow z\in d), a\in d\Rightarrow a\in c}}{\forall z(z\in c\leftrightarrow z\in d)\Rightarrow c=\{x:x\in d\}}\qquad \dfrac{a\in d\Rightarrow a\in d}{\Rightarrow d=\{x:x\in d\}}}{\forall z(z\in c\leftrightarrow z\in d)\Rightarrow c=d}}{\Rightarrow\forall z(z\in c\leftrightarrow z\in d)\to c=d}}{\Rightarrow\forall xy(\forall z(z\in x\leftrightarrow z\in y)\to x=y)}
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$$

Note that 2-premiss variant of LL was used to simplify a proof but to avoid the problems with cut-reduction we have to use 3-premiss version.

Application to set-builders

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Rules of abstraction (with stratified φ):

$$(Abs\Rightarrow)\ \frac{\varphi(t),\Gamma\Rightarrow\Delta}{t\in\{x:\varphi(x)\},\Gamma\Rightarrow\Delta}$$

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Application to set-builders

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are derivable by his rules. as well as (Ext) and (AV).

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How to formalize = to obtain cut-free system for NF?

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for φ atomic but not identity.

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Possible reductions in the application of $(\Rightarrow =)$:
at least two of t, t', t'' are complex.

Application to set-builders

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Possible reductions to applications of $(\Rightarrow=)$:

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Possible reductions to applications of ($\Rightarrow=$):

Consider the cases with at most one term t complex:

1. $a = b, a = c \vdash b = c$
2. $t = b, t = c \vdash b = c$
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the first rules may be modified to cover case 1 and 2:

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for $\varphi(t)$ atomic or atomic identity of the form $b = c$.

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For cases 3 and 4 we add rules:

$$\frac{\Gamma \Rightarrow \Delta, a = t \qquad t = c, \Gamma \Rightarrow \Delta}{a = c, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, a = t \qquad b = t, \Gamma \Rightarrow \Delta}{a = b, \Gamma \Rightarrow \Delta}$$

Methodological interlude

Methodological interlude

How do we build the rules - the case of $\exists_1$:

Methodological interlude

How do we build the rules - the case of $\exists_1$:

It may be defined in at least 3 equivalent ways:

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Using Rule-maker theorem (Indrzejczak [2013]):

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Using Rule-maker theorem (Indrzejczak [2013]):

For any sequent $\Gamma \Rightarrow \Delta$ with $\Gamma = \{\varphi_1, ..., \varphi_k\}$ and $\Delta = \{\psi_1, ..., \psi_n\}$, $k \geq 0$, $n \geq 0$ there is $2^{k+n} - 1$ equivalent rules captured by the general schema:

$$\frac{\Pi_1, \Rightarrow \Sigma_1, \varphi_1, \ \ldots, \ \Pi_i \Rightarrow \Sigma_i, \varphi_i \qquad \psi_1, \Pi_{i+1} \Rightarrow \Sigma_{i+1}, \ \ldots, \ \psi_j, \Pi_{i+j} \Rightarrow \Sigma_{i+j}}{\Gamma^{-i}, \Pi_1, ..., \Pi_i, \Pi_{i+1}, ..., \Pi_{i+j} \Rightarrow \Sigma_1, ..., \Sigma_i, \Sigma_{i+1}, ..., \Sigma_{i+j}\Delta^{-j}}$$

where $\Gamma^{-i} = \Gamma - \{\varphi_1, ..., \varphi_i\}$ and $\Delta^{-j} = \Delta - \{\psi_1, ..., \psi_j\}$ for $0 \leq i \leq k, \ 0 \leq j \leq n$.

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where $\Gamma^{-i} = \Gamma - \{\varphi_1, ..., \varphi_i\}$ and $\Delta^{-j} = \Delta - \{\psi_1, ..., \psi_j\}$ for $0 \leq i \leq k,\ 0 \leq j \leq n$.

We can replace any sequent with different interderivable (by structural rules only) rules.

Methodological interlude

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For example from: $\exists x \forall y(\varphi[x/y] \leftrightarrow y = x) \Rightarrow \exists_1 x\varphi$

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we can obtain two rules:

$$\frac{\Gamma \Rightarrow \Delta, \exists x \forall y(\varphi[x/y] \leftrightarrow y = x)}{\Gamma \Rightarrow \Delta, \exists_1 x\varphi}$$

and

$$\frac{\exists_1 x\varphi, \Gamma \Rightarrow \Delta}{\exists x \forall y(\varphi[x/y] \leftrightarrow y = x), \Gamma \Rightarrow \Delta}$$

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The first may be used as the basis for the introduction rule but still bad (no subformula-property, no separation).

Methodological interlude

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We continue with decomposition of side-formula $\exists x \forall y(\varphi[x/y] \leftrightarrow y = x)$ obtaining:

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We continue with decomposition of side-formula $\exists x \forall y (\varphi[x/y] \leftrightarrow y = x)$ obtaining:

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where a is not in Γ, Δ, φ

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One may test that it works by proving the corresponding sequent:

$$\dfrac{(\Rightarrow \exists_1)\ \dfrac{(\forall \Rightarrow)\ \dfrac{\varphi(b) \leftrightarrow b = a, \varphi(b) \Rightarrow b = a}{\forall y(\varphi(y) \leftrightarrow y = a), \varphi(b) \Rightarrow b = a} \qquad \dfrac{\varphi(b) \leftrightarrow b = a, b = a \Rightarrow \varphi(b)}{\forall y(\varphi(y) \leftrightarrow y = a), b = a \Rightarrow \varphi(b)}}{(\exists \Rightarrow)\ \forall y(\varphi(y) \leftrightarrow y = a) \Rightarrow \exists_1 x\varphi}}{\exists x \forall y(\varphi(y) \leftrightarrow y = x) \Rightarrow \exists_1 x\varphi}$$

Methodological interlude

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However, when we try the same with:

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The reason is that existentially and universally quantified variables occur in the same scope. So the method of decomposition does not yield the required result which allows us to prove definitional equivalences universally.

Methodological interlude

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The same situation holds for:
$\exists_1 x\varphi \Rightarrow \exists x(\varphi \wedge \forall y(\varphi[x/y] \rightarrow y = x))$ and
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they lead to the rules:

$$(\exists_1 \Rightarrow) \quad \frac{\varphi[x/a], \Gamma \Rightarrow \Delta, \varphi[x/b] \qquad b = a, \varphi[x/a], \Gamma \Rightarrow \Delta,}{\exists_1 x\varphi, \Gamma \Rightarrow \Delta}$$

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The second rule works but when we try to prove the first sequent by means of the first rule a derivation breaks.

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In general: to obtain a decent rule the quantifiers in decomposed formulae should have separate scopes.

Methodological interlude

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On the basis of:

$\exists_1 x\varphi \Rightarrow \exists x\varphi$

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We obtain the following three rules:

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists_1 x\varphi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi(b) \qquad \Gamma \Rightarrow \Delta, \varphi(c) \qquad b = c, \Gamma \Rightarrow \Delta}{\exists_1 x\varphi, \Gamma \Rightarrow \Delta}$$

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Of course, instead of 2- or 3-premise rules we can obtain rules with reduced branching-factor by RG-theorem, e.g:

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or

$$\frac{b = c, \Gamma \Rightarrow \Delta}{\varphi(b), \varphi(c), \exists_1 x\varphi, \Gamma \Rightarrow \Delta}$$

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or even:

$$\frac{\varphi(a), \varphi(b), \Gamma \Rightarrow \Delta, a = b}{\varphi(b), \Gamma \Rightarrow \Delta, \exists_1 x \varphi}$$

Methodological interlude

How do we build the rules - the case of $\exists_1$:

$$\frac{\Gamma \Rightarrow \Delta, \varphi(b) \qquad \varphi(a), \varphi(a'), \Gamma \Rightarrow \Delta, a = a'}{\Gamma \Rightarrow \Delta, \exists_1 x \varphi}$$

may be replaced with:

$$\frac{\varphi(a), \varphi(a'), \Gamma \Rightarrow \Delta, a = a'}{\varphi(b), \Gamma \Rightarrow \Delta, \exists_1 x \varphi}$$

or even:

$$\frac{\varphi(a), \varphi(b), \Gamma \Rightarrow \Delta, a = b}{\varphi(b), \Gamma \Rightarrow \Delta, \exists_1 x \varphi}$$

Warning: but such simplifications usually lead to failure of the cut elimination theorem.

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