Definite Descriptions in Modal and Temporal Logics

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Introduction

Definite descriptions (DDs) are term-forming expressions, e.g., ‘the x such that $\varphi(x)$’.

There is a lot of research on DDs in first-order languages, but the following lack good understanding:

1. How adding DDs affects *propositional modal languages*?
2. How to express *temporal* DDs and reason about them?

We will address both of these topics.

Plan

The presentation will be divided into *two parts*:

1. Hybrid Modal Operators for Definite Descriptions

2. Temporal References via Definite Descriptions

1. Hybrid Modal Operators for Definite Descriptions

Motivations

DDs, and referring expressions in general, provide a *convenient way of identifying objects* in information and knowledge base management systems, (Toman, Weddell)

- e.g., report answer to queries with DDs instead of obscure ids: "Synchronicity" by "The Police" vs $/guid/9202a8c04000641f8000000002f9e349$

Thus, DDs are used in description logics, where they take the form

$$\{\iota C\}$$

The extension of such an expression is the *unique a satisfying concept C* if it exists, or $\emptyset$ if there is no such a
(Artale, Mazzullo, Ozaki, Wolter).

Motivations

Properties of DDs in *specific description logics* have been studied

- e.g., it was shown that *nominals and universal roles can express* DDs (Artale, Mazzullo, Ozaki, Wolter).

However, the basic questions remain open:

- What is the *complexity cost* of adding DDs to a modal language?
- What new do DDs *allow us to express* in modal languages?

Modal Operators for DDs

- We introduce operators $@_{\iota\varphi}$, for any formula φ.
- $@_{\iota\varphi_1}\varphi_2$ is to mean that *'φ_2 holds in the unique world in which φ_1 holds'*.
- For example $@_{\iota CKF}Bald$ is to mean that 'the current king of France is bald' .

Our Results

Computational complexity (for satisfiability checking):

- $\mathcal{ML}(\iota)$-satisfiability is ExpTime-complete.
- $\mathcal{ML}(\iota)$-satisfiability is PSpace-complete if we allow for Boolean DDs only.

Relative expressiveness (existence of equivalence preserving translations):

- $\mathcal{H}(@) \prec \mathcal{ML}(\iota) \prec \mathcal{MLC}$ (arbitrary frames)
- $\mathcal{H}(@) \prec_L \mathcal{ML}(\iota) \prec_L \mathcal{MLC}$ (linear frames)
- $\mathcal{H}(@) \prec_{\mathbb{Z}} \mathcal{ML}(\iota) \approx_{\mathbb{Z}} \mathcal{MLC}$ (integer frame)

Syntax of $\mathcal{ML}(\iota)$

$\mathcal{ML}(\iota)$-formulas are generated as follows, where $p \in$ PROP:

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \Diamond\varphi \mid @_{\iota\varphi_1}\varphi_2,$$

($\top$, $\bot$, $\vee$, $\rightarrow$, $\Box$ are treated as the usual abbreviations)

We call $@_{\iota\varphi}$ a *definite description*; we call it *Boolean* if so is φ.

Semantics of $\mathcal{ML}(\iota)$

A *model* is a triple $\mathcal{M} = (W, R, V)$ with:

- $W \neq \emptyset$,
- $R \subseteq W \times W$,
- V : PROP $\longrightarrow \mathcal{P}(W)$.

Satisfaction of a formula in $\mathcal{M}$ and $w \in W$ is defined recursively:

$\mathcal{M}, w \models p$	iff	$w \in V(p)$, for each $p \in$ PROP
$\mathcal{M}, w \models \neg\varphi$	iff	$\mathcal{M}, w \not\models \varphi$
$\mathcal{M}, w \models \varphi_1 \vee \varphi_2$	iff	$\mathcal{M}, w \models \varphi_1$ or $\mathcal{M}, w \models \varphi_2$
$\mathcal{M}, w \models \Diamond\varphi$	iff	there exists $v \in W$ such that $(w, v) \in R$ and $\mathcal{M}, v \models \varphi$
$\mathcal{M}, w \models @_{\iota\varphi_1}\varphi_2$	iff	there exists a unique $v \in W$ such that $\mathcal{M}, v \models \varphi_1$ and moreover $\mathcal{M}, v \models \varphi_2$

Counting Logic $\mathcal{MLC}$

$\mathcal{MLC}$-formulas are generated by the following grammar, where $n \in \mathbb{N}$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \Diamond\varphi \mid \exists_{\geq n}\varphi,$$

- $\exists_{\leq n}\varphi$ abbreviates $\neg\exists_{\geq n+1}\varphi$
- $\exists_{=n}\varphi$ abbreviates $\exists_{\geq n}\varphi \wedge \exists_{\leq n}\varphi$

Additional condition:

$\mathcal{M}, w \models \exists_{\geq n}\varphi$ iff there are at least n worlds $v \in W$ such that $\mathcal{M}, v \models \varphi$

Hybrid Logic $\mathcal{H}(@)$

$\mathcal{H}(@)$-formulas are generated by the grammar

$$\varphi ::= p \mid i \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \Diamond\varphi \mid @_i\varphi_2,$$

where $i \in \text{NOM}$ are *nominals*.

Hybrid models $\mathcal{M} = (W, R, V)$ have $V : \text{PROP} \cup \text{NOM} \longrightarrow \mathcal{P}(W)$ assigning *singletons* to nominals.

Additional conditions:

$$\begin{array}{lll} \mathcal{M}, w \models i & \text{iff} & w \in V(i), \text{ for each } i \in \text{NOM} \\ \mathcal{M}, w \models @_i\varphi & \text{iff} & \mathcal{M}, v \models \varphi, \text{ for the unique } v \text{ such that } v \in V(i) \end{array}$$

Similarities

Basic relations between $\mathcal{ML}(\iota)$, $\mathcal{MLC}$, and $\mathcal{H}(@)$:

- We can express $@_{\iota\varphi_1}\varphi_2$ as $\exists_{=1}\varphi_1 \wedge \exists_{=1}(\varphi_1 \wedge \varphi_2)$.
- We can simulate a nominal i with a propositional variable p_i by writing $@_{\iota p_i}\top$.
- Then, we can express $@_i\varphi$ as $@_{\iota p_i}\varphi$.

Computational Complexity

Computational Complexity

Known results:

- $\mathcal{H}(@)$-satisfiability is PSpace-*complete* (Areces, Blackburn, Marx),
- $\mathcal{MLC}$-satisfiability is ExpTime-*complete* with unary encoded numbers (PhD of Tobies),
- $\mathcal{MLC}$-satisfiability is NExpTime-*complete* with binary encoded numbers (Zawidzki, Schmidt, Tishkovsky).

How does $\mathcal{ML}(\iota)$ fit into this picture?

New results:

- $\mathcal{ML}(\iota)$-satisfiability is ExpTime-*complete*,
- $\mathcal{ML}(\iota)$-satisfiability with Boolean DDs is PSpace-*complete*.

$\mathcal{ML}(\iota)$-satisfiability is ExpTime-complete

Proof.

For ExpTime-*hardness* reduce $\mathcal{ML}(\mathsf{A})$-satisfiability:

1. Transform $\mathcal{ML}(\mathsf{A})$-formula to NNF formula φ (it will mention $\wedge$, $\Box$, E),
2. Let $\varphi' = \neg s \wedge @_{\iota s}\top \wedge @_{\iota \Diamond s} s \wedge \tau(\varphi)$,
3. Where τ translates $\mathcal{ML}(\mathsf{A})$-formulas in NNF to $\mathcal{ML}(\iota)$-formulas:

$$\begin{aligned} \tau(p) &= p, & \tau(\Diamond\psi) &= \Diamond\tau(\psi), \\ \tau(\neg p) &= \neg p, & \tau(\Box\psi) &= \Box\tau(\psi), \\ \tau(\psi \vee \chi) &= \tau(\psi) \vee \tau(\chi), & \tau(\mathsf{E}\psi) &= @_{\iota p_\psi}(\tau(\psi) \wedge \neg s), \\ \tau(\psi \wedge \chi) &= \tau(\psi) \wedge \tau(\chi), & \tau(\mathsf{A}\psi) &= @_{\iota(s \vee \neg\tau(\psi))}\top, \end{aligned}$$

4. Show that *φ and φ' are equisatisfiable*.

□

Corollary: Modal logic with $\exists_{=1}$ is ExpTime-complete.

Game

For a formula φ, *define the following game*:

In the first turn *Eloise plays a set $\mathcal{H}$* of at most $|\iota(\varphi)| + 1$ (for $\iota(\varphi)$ being the set of formulas ψ such that $@_{\iota\psi}$ occurs in φ) Hintikka sets and $R \subseteq \mathcal{H} \times \mathcal{H}$ such that:

- $\varphi \in H$, for some $H \in \mathcal{H}$,
- each $\psi \in \iota(\varphi)$ can occur in at most one $H \in \mathcal{H}$,
- for all $@_{\iota\psi}\chi \in \mathsf{cl}(\varphi)$ and $H \in \mathcal{H}$ we have $@_{\iota\psi}\chi \in H$ iff there is $H' \in \mathcal{H}$ such that $\{\psi, \chi\} \subseteq H'$,
- and for all $\Diamond\psi \in \mathsf{cl}(\varphi)$, if $R(H, H')$ and $\psi \in H'$, then $\Diamond\psi \in H$.

Game

Abelard selects $H \in \mathit{Current}$ (initially $\mathit{Current} = \mathcal{H}$) *and a formula* $\Diamond\varphi' \in H$ (modal depth of chosen formula needs to decrease in each turn).

Eloise plays a Hintikka set H' such that

- $\varphi' \in H'$,
- if $H' \cap \iota(\varphi) \neq \emptyset$, then $H' \in \mathcal{H}$,
- for all $@_{\iota\psi}\chi \in \mathrm{cl}(\varphi)$ we have $@_{\iota\psi}\chi \in H'$ iff there is $H'' \in \mathcal{H}$ such that $\{\psi, \chi\} \subseteq H''$,
- and for all $\Diamond\psi \in \mathrm{cl}(\varphi)$, if $\psi \in H'$, then $\Diamond\psi \in H$.

If $H' \cap \iota(\varphi) \neq \emptyset$, then Eloise wins.

Otherwise, it is Abelard's turn with $\mathcal{H} ::= \mathcal{H} \cup \{H'\}$ and $\mathit{Current} ::= \{H'\}$.

If a player cannot make a move, they lose.

$\mathcal{ML}(\iota)$-satisfiability with Boolean DDs is PSpace-complete.

Proof.

Let φ be an $\mathcal{ML}(\iota)$-formula φ with Boolean DDs.

1. φ is satisfiable iff Eloise has a winning strategy in the game,
2. Game depth is polynomial and so are representations of game states,
3. Hence, the existence of a winning strategy is in PSpace (as AP $=$ PSpace, by Chandra-Kozen-Stockmeyer Theorem).

□

Expressive Power

Expressive Power

New results:

- $\mathcal{H}(@) \prec \mathcal{ML}(\iota) \prec \mathcal{MLC}$
- $\mathcal{H}(@) \prec_L \mathcal{ML}(\iota) \prec_L \mathcal{MLC}$
- $\mathcal{H}(@) \prec_{\mathbb{Z}} \mathcal{ML}(\iota) \approx_{\mathbb{Z}} \mathcal{MLC}$

where L stands for strict linear frames, and $\mathbb{Z}$ for the ordered set of integers $(\mathbb{Z}, <)$.

Bisimulation form $\mathcal{ML}(\iota)$

Definition. A *ι-bisimulation* between $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ is any *total (i.e., serial and surjective)* $Z \subseteq W \times W'$ such that whenever $(w, w') \in Z$:

- Atom: w and w' satisfy the same propositional variables,
- Zig: if there is $v \in W$ such that $(w, v) \in R$, then there is $v' \in W'$ such $(v, v') \in Z$ and $(w', v') \in R'$,
- Zag: if there is $v' \in W'$ such that $(w', v') \in R'$, then there is $v \in W$ such $(v, v') \in Z$ and $(w, v) \in R$,
- Singular: *$Z(w) = \{w'\}$ if and only if $Z^{-1}(w') = \{w\}$.*

Bisimulation form $\mathcal{ML}(\iota)$

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- Atom: w and w' satisfy the same propositional variables,
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- Zag: if there is $v' \in W'$ such that $(w', v') \in R'$, then there is $v \in W$ such $(v, v') \in Z$ and $(w, v) \in R$,
- Singular: *$Z(w) = \{w'\}$ if and only if $Z^{-1}(w') = \{w\}$.*

Lemma (Bisimulation Invariance). If there is a ι-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ which maps w to w', then

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}', w' \models \varphi$$

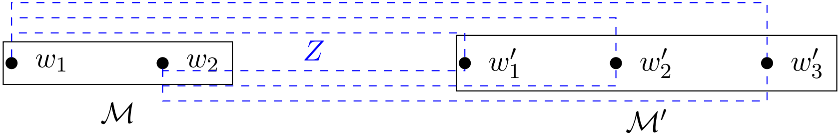
for any $\mathcal{ML}(\iota)$-formula φ.

$\mathcal{ML}(\iota) \prec \mathcal{MLC}$

Proof.

There is no $\mathcal{ML}(\iota)$-formula equivalent to $\mathcal{MLC}$-formula $\exists_{=2}\top$.

Indeed, consider models $\mathcal{M}$ and $\mathcal{M}'$ and a ι-bisimulation between them:

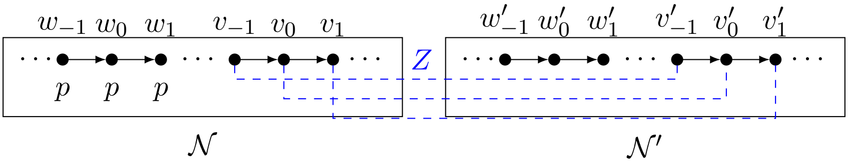


□

$\mathcal{ML}(\iota) \prec_L \mathcal{MLC}$

Proof.

There is no $\mathcal{ML}(\iota)$-formula equivalent to $\mathcal{MLC}$-formula $\exists_{\geq 1} p$ over linear frames.



1. $\mathcal{N}, v_0 \models \exists_{\geq 1} p$, but $\mathcal{N}', v'_0 \not\models \exists_{\geq 1} p$.
2. Z is a (standard) bisimulation, so v_0 and v'_0 satisfy the same basic modal formulas.
3. Define ι-bisimulation $Z_{\mathcal{N}} = \{(w_n, w_m), (v_n, v_m) \mid n, m \in \mathbb{Z}\}$ between $\mathcal{N}$ and $\mathcal{N}$.
4. So all w_n and all v_m satisfy the same $\mathcal{ML}(\iota)$-formulas in $\mathcal{N}$ (and in $\mathcal{N}'$).
5. So no DD is proper, and so, v_0 and v'_0 satisfy the same $\mathcal{ML}(\iota)$-formulas.

□

$\mathcal{ML}(\iota) \approx_{\mathbb{Z}} \mathcal{MLC}$

Proof.

1. Let $\psi_n = \psi \wedge \Diamond(\psi \wedge \Diamond(\psi \wedge \dots))$, where ψ occurs n times.
2. Show that $\exists_{\geq n}\psi$ *is equivalent to* $\Diamond\psi_n \vee @_{\iota(\psi_n \wedge \neg\Diamond\psi_n)}\top$ over $\mathbb{Z}$.
3. Indeed, $\exists_{\geq n}\psi$ holds at w if either

   (1) there are $w_1 < \dots < w_n$, all larger than w, in which ψ holds or

   (2) there exists the unique w' such that ψ holds in w' and in exactly $n-1$ words larger than w'.

   (1) is expressed by $\Diamond\psi_n$ and *(2) by* $@_{\iota(\psi_n \wedge \neg\Diamond\psi_n)}\top$.

□

Conclusions for Part 1

Complexity:

- $\mathcal{ML}(\iota)$-satisfiability is ExpTime-*complete* (like $\mathcal{MLC}$ with unary encoded numbers),
- $\mathcal{ML}(\iota)$-satisfiability with Boolean DDs is PSpace-*complete* (like $\mathcal{H}(@)$ and basic modal logic).

Expressiveness:

- $\mathcal{H}(@) \prec \mathcal{ML}(\iota) \prec \mathcal{MLC}$
- $\mathcal{H}(@) \prec_L \mathcal{ML}(\iota) \prec_L \mathcal{MLC}$
- $\mathcal{H}(@) \prec_{\mathbb{Z}} \mathcal{ML}(\iota) \approx_{\mathbb{Z}} \mathcal{MLC}$

2. Temporal References via Definite Descriptions

Motivations

Referring to *particular points of time* is essential for our everyday communication and for knowledge representation systems, e.g., consider

- ‘the last time I read "On Denoting"’,
- ‘the time when the system was upgraded to version 2.0’.

Temporal reference can be:

1. *definite*, when we refer to a unique point of time
   - e.g., in past simple tenses:
     “I didn’t turn off the stove”
   - corresponds to the article ‘the’
2. *indefinite*, otherwise
   - e.g., in present perfect tenses:
     “Have you ever eaten caviar before?”, “No. But I have eaten oysters”
   - corresponds to the article ‘a’.

We will focus on type 1.

Related Work

- *Tense Logic*: tense operators to express temporal relations between time points, (Łoś, Prior, von Wright, etc.)
- FO($<$): temporal logics are related to FO:
  - FOMLO is TL(U,S) (Kamp)
  - $\text{FO}^2(<)$ is unary-TL (Etessami, Vardi, Wilke)
- *Hybrid Logics*: use clock-variables (Prior) a.k.a. names (Gargov, Goranko) a.k.a. nominals (Blackburn) to label points of a model
- *First-order Logics*: use term-forming operators, e.g., ι operator (Peano)
- *Context of reference*: time of utterance is a crucial component of the context, model it with a two-dimensional logic (Kamp) or a special constant now (Prior)

Contributions

1. *Logic* for expressing complex temporal references,
2. *Sound and complete* tableau system for the logic,
3. *Complexity* results for well-behaving fragments of the logic,

FO$(<, \iota, now)$

We obtain FO$(<, \iota, now)$ by extending first-order monadic logic of order FO$(<)$ with:

- operator ι, for DDs
- constant now, for the time of utterance.

FO$(<, \iota, now)$ allows us to express *complex temporal references*, for example the term

'the last time I met Mary'

$$\iota x\Big(MeetM(x) \land x < now \land \forall y(x < y < now \to (\neg MeetM(y))\Big).$$

Syntax of FO$(<, \iota, now)$

Vocabulary:

- set Σ of unary predicates $P, Q, R, \ldots,$
- set VAR of first-order variables $x, y, z, \ldots,$
- $\neg$, $\vee$, $\exists$,
- *earlier-later relation* $<$,
- *definite description operator* ι,
- *constant* now.

Terms s and *formulas* φ are defined simultaneously:

$$s ::= x \mid now \mid \iota x \varphi(x)$$

$$\varphi ::= P(s) \mid s_1 = s_2 \mid s_1 < s_2 \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists x \varphi(x),$$

- $\varphi(x)$ is a formula with a free variable x, where variables are bound by both quantifiers and the ι-operator.

Syntax of FO$(<, \iota, now)$

'I was studying "On Denoting" the last time I met Mary; therefore, I have not met her since I was studying "On Denoting"':

$$Study(s_M) \to \exists x \Big(x < now \wedge Study(x) \wedge \forall y(x < y < now \to \neg MeetM(y)) \Big),$$

where $s_M ::= \iota x \Big(MeetM(x) \wedge x < now \wedge \forall y(x < y < now \to (\neg MeetM(y)) \Big).$

FO$^2(<, \iota, now)$ is the 2-variable fragment of FO$(<, \iota, now)$. It enables to express, e.g., 'I have not met John since I met Mary'

$$\exists x \Big(x < now \wedge MeetM(x) \wedge \forall y(x < y < now \to \neg MeetJ(y)) \Big).$$

FO$(<, \iota, now)$ semantics

Model is a tuple $\mathcal{M} = (\mathcal{T}, <, \mathcal{I}, t_0)$ where

- $\mathcal{T}$ is a set of of time points (strictly) linearly ordered by $<$,
- $\mathcal{I} : \Sigma \longrightarrow \mathcal{P}(\mathcal{T})$
- $t_0 \in \mathcal{T}$ is the time of utterance.

$\mathcal{M}, v \models \varphi$, for assignment v of constants, is defined as usual for φ with no ι-operators.

For ι-operators we adopt the *Russellian semantics*:

$$\mathcal{M}, v \models P(\iota x \varphi(x)) \quad \text{iff} \quad \text{there exists a unique } t \in \mathcal{T} \text{ such that } \mathcal{M}, v[x \mapsto t] \models \varphi(x), \text{ and } t \in \mathcal{I}(P) \text{ for this } t$$

$$\mathcal{M}, v \models s \leqslant \iota x \varphi(x) \quad \text{iff} \quad \text{there exists a unique } t \in \mathcal{T} \text{ such that } \mathcal{M}, v[x \mapsto t] \models \varphi(x), \text{ and } v(s) \leqslant t \text{ for this } t$$

$$\mathcal{M}, v \models \iota x \varphi(x) \leqslant \iota y \psi(y) \quad \text{iff} \quad \text{there exist unique } t_1, t_2 \in \mathcal{T} \text{ such that } \mathcal{M}, v[x \mapsto t_1] \models \varphi(x) \text{ and } \mathcal{M}, v[y \mapsto t_2] \models \psi(y), \text{ and moreover } t_1 \leqslant t_2 \text{ for these } t_1, t_2$$

FO$(<, \iota, now)$ semantics

Note that $s_1 = s_2$ *is not equivalent to* $\neg(s_1 < s_2) \wedge \neg(s_1 > s_2)$,

e.g., if s_1 or s_2 is an improper DD, then $s_1 = s_2$ is not true, but the latter is true.

A formula is *valid* if it is satisfied in every model and every assignment, e.g.,

'I was studying "On Denoting" the last time I met Mary; therefore, I have not met her since I was studying "On Denoting"':

$$Study(s_M) \rightarrow \exists x\Big(x < now \wedge Study(x) \wedge \forall y(x < y < now \rightarrow \neg MeetM(y))\Big),$$

Tableau System

A *tableau-proof* of φ is any closed tableau with $\neg\varphi$ at the root.

We use the following convention:

- s, s_1, s_2 are terms,
- d, d_1, d_2 are DDs,
- $x \in \mathsf{VAR}$,
- $a, a_1, a_2 \in \mathsf{VAR}$ are free and freshly introduced to the branch by a rule application,
- $b, b_1, b_2, b_3 \in \mathsf{VAR} \cup \{now\}$ are free and need to be present on a branch before a rule application,
- $\varphi[s_1/s_2]$ is φ with all occurrences of s_1 substituted by s_2,
- $\varphi[s_1//s_2]$ is φ with some occurrences of s_1 substituted by s_2.

Tableau System

Timeline rules:

$$(\mathsf{NE})^* \ \frac{}{a = a} \qquad (\mathsf{tran}) \ \frac{b_1 < b_2, b_2 < b_3}{b_1 < b_3} \qquad (\mathsf{irref}) \ \frac{b < b}{\bot} \qquad (\mathsf{trich}) \ \frac{}{b_1 < b_2 \mid b_1 = b_2 \mid b_2 < b_1}$$

Basic first-order rules:

$$(\neg\neg) \ \frac{\neg\neg\varphi}{\varphi} \qquad (\vee) \ \frac{\varphi \vee \psi}{\varphi \mid \psi} \qquad (\neg\vee) \ \frac{\neg(\varphi \vee \psi)}{\begin{array}{c}\neg\varphi \\ \neg\psi\end{array}} \qquad (\exists) \ \frac{\exists x \varphi}{\varphi[x/a]} \qquad (\neg\exists) \ \frac{\neg\exists x \varphi}{\neg\varphi[x/b]}$$

$$(\mathsf{sym}) \ \frac{s_1 = s_2}{s_2 = s_1} \qquad (\mathsf{rep}) \ \frac{s_1 = s_2, \varphi(s_1)}{\varphi[s_1 // s_2]} \qquad (\mathsf{clash}) \ \frac{\varphi, \neg\varphi}{\bot}$$

Definite description rules:

$$(\iota\mathsf{S}_1) \ \frac{P(d)}{a = d} \qquad (\iota\mathsf{S}_2) \ \frac{d < s}{a = d} \qquad (\iota\mathsf{S}_3) \ \frac{s < d}{a = d} \qquad (\iota\mathsf{S}_4) \ \frac{d_1 = d_2}{a = d_1}$$

$$(\iota\mathsf{E}_1) \ \frac{b = \iota x \varphi(x)}{\varphi[x/b]} \qquad (\iota\mathsf{E}_2) \ \frac{b_1 = \iota x \varphi(x)}{\neg\varphi[x/b_2] \mid b_1 = b_2} \qquad (\neg\iota\mathsf{E}) \ \frac{b \neq \iota x \varphi(x)}{\neg\varphi[x/b] \ \Big| \begin{array}{c} a \neq b \\ \varphi[x/a]\end{array}} \qquad (\mathsf{cut}) \ \frac{}{b = d \mid b \neq d}$$

* (NE) can be applied only if there are no free variables or now on the branch and no other rules are applicable.

Example of a Tableau-proof

We show a proof for 'I was studying "On Denoting" the last time I met Mary; therefore, I have not met her since I was studying "On Denoting"':

$$\neg\Big(Study(s_M) \to \exists x(x < now \wedge Study(x) \wedge \forall y(x < y \wedge y < now \to \neg MeetM(y)))\Big)$$

$$\downarrow(\neg\to)$$

$$Study(s_{LastM})$$

$$\neg\exists x(x < now \wedge Study(x) \wedge \forall y(x < y \wedge y < now \to \neg MeetM(y)))$$

$$\downarrow(\iota \mathsf{S}_1)$$

$$a = s_{LastM}$$

$$\downarrow(\mathsf{rep})$$

$$Study(a)$$

$$\downarrow(\neg\exists)$$

$$\neg(a < now \wedge Study(a) \wedge \forall y(a < y \wedge y < now \to \neg MeetM(y)))$$

$$\downarrow(\iota \mathsf{E}_1)$$

$$MeetM(a) \wedge a < now \wedge \forall y(a < y \wedge y < now \to \neg MeetM(y))$$

Example of a Tableau-proof Cont.

$$MeetM(a) \wedge a < now \wedge \forall y(a < y \wedge y < now \rightarrow \neg MeetM(y))$$

$$\downarrow (\wedge)$$

$$MeetM(a)$$

$$a < now$$

$$\forall y(a < y \wedge y < now \rightarrow \neg MeetM(y))$$

$$(\neg\wedge)$$

$$\neg a < now \quad \text{(clash)} \downarrow \quad \bot$$

$$\neg(Study(a) \wedge \forall y(a < y \wedge y < now \rightarrow \neg MeetM(y)))$$

$$(\neg\wedge)$$

$$\neg Study(a) \quad \text{(clash)} \downarrow \quad \bot$$

$$\neg\forall y(a < y \wedge y < now \rightarrow \neg MeetM(y))$$

$$\downarrow (\neg\forall)$$

$$\neg(a < a' \wedge a' < now \rightarrow \neg MeetM(a'))$$

$$\downarrow (\forall)$$

$$a < a' \wedge a' < now \rightarrow \neg MeetM(a')$$

$$\downarrow \text{(clash)}$$

$$\bot$$

Soundness

Calculus is *sound* if every FO$(<, \iota, now)$-formula φ that has a tableau-proof is valid.

Theorem. Our calculus is sound.

Proof.

1. Show the *Coincidence Lemma*, i.e., $\mathcal{M}, v_1 \models \varphi$ iff $\mathcal{M}, v_2 \models \varphi$, for any v_1, v_2 agreeing on free variables in φ.

2. Show the *Substitution Lemma*, i.e., $\mathcal{M}, v \models \varphi[x/a]$ iff $\mathcal{M}, v[x \mapsto v(a)] \models \varphi$, for x and a free variables in φ.

3. Show that for each rule $\dfrac{\Phi}{\Psi_1 \mid \ldots \mid \Psi_n}$, *if Φ is satsifiable, then so is some Ψ_i*.

□

Completeness

Calculus is *complete* if every FO($<, \iota, now$)-formula φ that is valid has a tableau-proof.

Theorem. Our calculus is complete.

Proof.

1. Let $\mathcal{B}$ be an open, expanded branch with root $\neg\varphi$; to show that $\neg\varphi$ is satisfiable.
2. Let $b_1 \approx b_2$ iff $b_1 = b_2$ is in $\mathcal{B}$.
3. Construct $\mathcal{M} = (\mathcal{T}, <, \mathcal{I}, t_0)$ and $v : \text{VAR} \longrightarrow \mathcal{T}$ such that:

$$\mathcal{T} = \{[b]_\approx \mid b \in \text{TERM}\}, \qquad < = \{([b_1]_\approx, [b_2]_\approx) \in \mathcal{T} \times \mathcal{T} \mid b_1 < b_2 \in \mathcal{B}\},$$

$$\mathcal{I}(P) = \{[s]_\approx \in \mathcal{T} \mid P(s) \in \mathcal{B}\}, \qquad \mathcal{I}(now) = t_0,$$

$$t_0 = \begin{cases} [now]_\approx & \text{if } now \text{ occurs on } \mathcal{B}, \\ [b_0]_\approx & \text{otherwise,} \end{cases} \qquad v(x) = \begin{cases} [x]_\approx & \text{if } x \text{ is free on } \mathcal{B}, \\ t_0 & \text{otherwise,} \end{cases}$$

4. Show by induction on the structure of ψ that $\psi \in \mathcal{B}$ implies $\mathcal{M}, v \models \psi$.

Complexity

Theorem. Satisfiability checking over $\mathbb{N}$ is:

- *decidable*, but not elementary recursive in FO$(<, \iota, now)$,
- NExpTime-*complete* in FO$^2(<, \iota, now)$,
- NP-*complete* in FO$^2(<, \iota, now)$ with bounded number of predicates.

Proof.

1. It suffices to show that each formula of FO$(<, \iota, now)$ (resp. FO$^2(<, \iota, now)$) can be polynomially translated to an *equisatisfiable formula of* FO$(<)$ *(resp.* FO$(<)^2$*)*.
2. Trick: x in $\iota x \psi(x)$ is not free, so *rewrite x with different variable*, e.g.

$$\tau(P(\iota x \psi(x))) = \exists z (P(z) \land \forall x (\tau(\psi(x)) \leftrightarrow x = z))$$
$$\tau(x \lesseqgtr \iota y \psi(y)) = \exists z (x \lesseqgtr z \land \forall x (\tau(\psi(x)) \leftrightarrow x = z))$$

□

Conclusions for Part 2

- FO$(<, \iota, now)$ is a dedicated language for complex *temporal references* thanks to
  - exploiting ι for temporal reference
  - capturing temporal context with now
- FO$(<, \iota, now)$ has a *sound an complete* tableau system,
- Reasoning in FO$(<, \iota, now)$ is *decidable* with NExpTime- and NP-*complete* fragments.

Future directions

- DDs in first-order temporal logics
- DDs in two-dimensional temporal logics
- DDs in description logics

Thank you for your attention

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