A survey of systems formalising Definite Descriptions by Binary Quantification

Nils Kürbis

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Definite Descriptions via Binary Quantification

Definite descriptions are normally treated as term forming operations: 'the' takes a (simple or complex) predicate F and forms a singular term out of it: the F. 'The F' then takes a further predicate or further singular terms and a relational expression to form a sentence: 'The F is G' or 'a is R to the F'.

Formalising DD by the ι operator follows this pattern: ι binds a variable and forms a singular term from an open formula. This method is due to Peano and used by Whitehead and Russell, Hilbert and Bernays, Hintikka, Lambert, van Fraassen and others.

There is an alternative method that formalises the complete sentences in which DD occur in one go. 'The F is G' is formalised by a binary quantifier I that binds a variable and takes two open formulas to form a formula: $Ix[F, G]$.

This takes on Russell's idea that DD only appear to be singular terms and have meaning only in the context of complete sentences, and the view that DD are quantificational devices.

Plan of this Talk

I will give an overview over a number of options of rules of inference for the binary quantifier:

(1) As part of a system that stays close to Neil Tennant's system of natural deduction for intuitionist negative free logic with ι.
(2) These rules also work for classical negative free logic, but I'll rephrase them for sequent calculus.
(3) Rather complicated rules for positive free logic with the motivation of saying very little about improper DD, both intuitionist and classical, in natural deduction and sequent calculus.
(4) Greatly simplified rules for classical and intuitionist positive free logic, in natural deduction and sequent calculus again.

(3) and (4) are a contribution to formalising new theories of definite descriptions.

INF is intuitionist negative free logic, IPF is intuitionist positive free logic, both in natural deduction; CNF and CPF are their classical versions, both in sequent calculus.

Quantifiers in Free Logic. Natural Deduction.

The rules for the sentential connectives are as for intuitionist logic. Free logic alters the rules for the quantifiers and appeals to a primitive predicate ∃!, interpreted as 'exists' or 'refers':

$$(\forall I)\quad \dfrac{\begin{array}{c}[\exists! a]^i \\ \Pi \\ A^x_a\end{array}}{\forall x A}\, i \qquad\qquad (\forall E)\quad \dfrac{\forall x A \qquad \exists! t}{A^x_t}$$

$\forall I$: a not in A or any open assumptions of Π except $\exists! a$.

$$(\exists I)\quad \dfrac{A^x_t \qquad \exists! t}{\exists x A} \qquad\qquad (\exists E)\quad \dfrac{\exists x A \qquad \begin{array}{c}\underbrace{[A^x_a]^i,\ [\exists! a]^j} \\ \Pi \\ C\end{array}}{C}\, i,j$$

$\exists E$: a not in A, C or any open assumptions of Π except A^x_a, $\exists! a$.

Identity and Atomic Denotation. Natural Deduction

The elimination rule for identity is the same in positive and negative free logic. In the former, identity is also governed by the Law of Self-Identity:

$$= I: \quad \frac{}{t = t} \qquad\qquad = E: \quad \frac{t_1 = t_2 \qquad A^x_{t_1}}{A^x_{t_2}}$$

where A is an atomic formula.

In negative free logic, self-identity is conditional on $\exists! t$:

$$= I^n \, \frac{\exists! t}{t = t}$$

Atomic Denotation is characteristic of negative free logic. An atomic formula can only be true if all terms occurring in it refer:

$$AD \, \frac{Rt_1 ... t_n}{\exists! t_i}$$

for $1 \leq i \leq n$.

Free Logic in Sequent Calculus

Andrzej Indrzejczak cut free formalisation of positive free logic:

$$(L\forall)\quad \frac{A^x_t, \Gamma \Rightarrow \Delta}{\exists! t, \forall x A, \Gamma \Rightarrow \Delta} \qquad (R\forall)\quad \frac{\exists! a, \Gamma \Rightarrow \Delta, A^x_a}{\Gamma \Rightarrow \Delta, \forall x A}$$

$$(L\exists)\quad \frac{\exists! a, A^x_a, \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta} \qquad (R\exists)\quad \frac{\Gamma \Rightarrow \Delta, A^x_t}{\exists! t, \Gamma \Rightarrow \Delta, \exists x A}$$

where in $(L\exists)$ and $(R\forall)$, a does not occur in the conclusion.

$$(= I)\quad \frac{A^x_{t_2}, \Gamma \Rightarrow \Delta}{t_1 = t_2, A^x_{t_1}, \Gamma \Rightarrow \Delta} \qquad (= E)\quad \frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where A is atomic.

For negative free logic, replace $(= I)$ by (NEI) and add (NEE):

$$(NEI)\quad \frac{t = t, \Gamma \Rightarrow \Delta}{\exists! t, \Gamma \Rightarrow \Delta} \qquad (NEE)\quad \frac{\exists! t_1, \Gamma \Rightarrow \Delta}{R t_1 \ldots t_n, \Gamma \Rightarrow \Delta}$$

Rules for I in Intuitionist Negative Free Logic

$$II: \quad \dfrac{F^x_t \qquad G^x_t \qquad \exists! t \qquad \begin{matrix} [F^x_a]^i \; [\exists! a]^j \\ \Pi \\ a = t \end{matrix}}{Ix[F, G]} \; i,j$$

where t is free for x in F and in G, and a does not occur in t nor in any undischarged assumptions in Π except F^x_a and $\exists! a$.

$$IE^1: \quad \dfrac{Ix[F, G] \qquad \begin{matrix} [F^x_a]^i \; [G^x_a]^j \; [\exists! a]^k \\ \Pi \\ C \end{matrix}}{C} \; i,j,k$$

where a is not free in F, G, C nor any undischarged assumptions in Π except F^x_a, G^x_a and $\exists! a$.

$$\iota E^{2A}: \quad \dfrac{\iota x[F, G] \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad F^x_{t_2} \qquad A^x_{t_1}}{A^x_{t_2}}$$

where t_1 and t_2 are free for x in F and A is atomic.

Properties of I

Let INF^I be intuitionist negative free logic extended by the rules for the binary quantifier I. The following hold:

$*1.\ Ix[F, G] \vdash \exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$
$*2.\ \exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G) \vdash Ix[F, G]$
$*3.\ Ix[F, x = t], G^x_t \vdash Ix[F, G]$
$*4.\ \forall y(Ix[A, x = y] \leftrightarrow \forall x(A \leftrightarrow x = y))$
$*5.\ \exists!t \vdash Ix[x = t, x = t]$
$*6.$ Deductions in INF^I normalise

$*1$ and $*2$: $Ix[F, G]$ captures Russell's analysis of 'The F is G'.
$*3$ and variants thereof show that we can operate with formulas $Ix[F, G]$ as we would with terms in Leibniz' Law.
$*4$: analogue to Lambert's Axiom: $\forall y(\iota xA = y \leftrightarrow \forall x(A \leftrightarrow x = y))$
$*5$: analogue to the negative version of Lambert's second axiom.
$*6$ is a desirable proof theoretic property.

Tennant's Rules for ι

Tennant adds the following rules to INF. Call the system INF$^\iota$.

$$\iota I : \quad \dfrac{\exists! t \qquad \begin{array}{c}[a = t]^i \\ \Xi \\ F^x_a\end{array} \qquad \begin{array}{c}[F^x_a]^i \; [\exists! a]^i \\ \Pi \\ a = t\end{array}}{\iota x F = t}\, i$$

where a is not free in F and does not occur in any undischarged assumptions in Ξ and Π except those displayed.

$$\iota E^1 : \quad \dfrac{\iota x F = t \qquad u = t}{F^x_u} \qquad \iota E^2 : \quad \dfrac{\iota x F = t \qquad F^x_u \qquad \exists! u}{u = t}$$

$$\iota E^3 : \quad \dfrac{\iota x F = t}{\exists! t}$$

where u is free for x in F.

Comparison between INF^I and INF^ι

A direct comparison between the two systems is not possible: The binary quantifier I permits the drawing of scope distinctions. For instance, it is possible to distinguish between external and internal negation in INF^I: $\neg Ix[F, G]$ vs. $Ix[F, \neg G]$. There are no scope markers in INF^ι, so this distinction cannot be drawn and there is only the one formula $\neg G(\iota xF)$.

INF^I is more expressive than INF^ι, so we need to restrict it. The following will do: let INF^{IR} be like INF^I except that the G in $Ix[F, G]$ is restricted to identity. Analogously for $\mathsf{INF}^{\iota R}$ (in a way nothing is lost here, as rules of INF^ι do not tell us how to operate with formulas $G(\iota xF)$: we always need an identity). Then we have:

$\Gamma \vdash A$ in INF^{IR} iff $\tau(\Gamma) \vdash \tau(A)$ in $\mathsf{INF}^{\iota R}$, where τ translates $Ix[F, x = t]$ as $\iota xF = t$.

Tennant's rules are Lambert's Axiom in rule form, a formula that translates as Lambert's Axiom under τ is derivable in INF^I.

I in Classical Negative Free Logic

The following are appropriate rules for the binary quantifier I in negative free logic:

$$(RI) \quad \frac{\Gamma \Rightarrow \Delta, F^x_t \qquad \Gamma \Rightarrow \Delta, G^x_t \qquad F^x_a, \Gamma \Rightarrow \Delta, a = t}{\exists! t, \Gamma \Rightarrow \Delta, Ix[F, G]}$$

$$(LI^1) \quad \frac{F^x_a, G^x_a, \exists! a, \Gamma \Delta}{Ix[F, G], \Gamma \Delta}$$

$$(LI^2) \quad \frac{\Gamma \Rightarrow \Delta, F^x_{t_1} \qquad \Gamma \Rightarrow \Delta, F^x_{t_2} \qquad \Gamma \Rightarrow \Delta, A^x_{t_2}}{Ix[F, G], \exists! t_1, \exists! t_2, \Gamma \Rightarrow \Delta, A_{t_1}}$$

where in (RI) and (LI^1), a does not occur in the conclusion, and in (LI^2) A is an atomic formula.

Added to Indrzejczak's cut free formalisation of classical negative free logic results in a system CPF^I that is also cut free.

Negative vs. Positive Free Logic

The equivalence of $Ix[F, G]$ with $\exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)$ means that adding I to INF does not actually increase its expressiveness: I is definable. Negative free logic is very Russellian in this respect. (In INF$^\iota$, $G(\iota x F)$ is equivalent to the Russellian analysis if G is atomic.)

This situation is different in positive free logic. Here the idea is that atomic sentences can be true even if some terms occurring in them, including definite descriptions, do not refer. So the Russellian equivalence should not hold anymore.

The proof that shows that $\exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)$ and $Ix[F, G]$ are equivalent does not appeal to the law of self-identity or atomic denotation. Hence it also holds in positive free logic, if the rules for I we have considered so far were added to IPF or CPF.

The spirit of positive free logic asks for different rules for the binary quantifier.

The Binary Quantifier in Positive Free Logic

In positive free logic with a term forming ι operator, we have:

$*7.\ G(\iota x F) \wedge \exists! \iota x F \leftrightarrow \exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)$

To find rules for I suitable for positive free logic, I propose to assume that the analogous equivalence holds:

$*8.\ Ix[F, G] \wedge Ix[F, \exists! x] \leftrightarrow \exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)$

Exploiting these equivalences and casting them into rule form gives rather complicated rules for I, to be given on the next slide.

The resulting logic is also very weak. Hintikka and Lambert both declare that positive free logic should be largely silent on improper definite descriptions. With the rules to be given, we remain very quiet indeed about them: not even the analogue of the law of self-identity 'The $F =$ the F', i.e. $Ix[F, Iy[F^x_y, x = y]]$, holds.

Natural Deduction for the Binary Quantifier I for IPF

$$II: \quad \cfrac{F^x_t \qquad G^x_t \qquad \exists! t \qquad \begin{array}{c}\underbrace{[F^x_a]^i,\ [\exists! a]^j}\\ \Pi \\ a = t\end{array}}{Ix[F, G]}\ i,j$$

where a is different from t, does not occur in F, G or in any undischarged assumption in Π except F^x_a and $\exists! a$.

$$IE^{1p}: \quad \cfrac{Ix[F, G] \qquad F^x_t \qquad \exists! t \qquad \begin{array}{c}\underbrace{[F^x_a]^{i_1},\ [\exists! a]^{i_2}}\\ \Pi \\ a = t\end{array} \qquad \begin{array}{c}\underbrace{[F^x_b]^{i_3},\ [G^x_b]^{i_4},\ [\exists! b]^{i_5}}\\ \Sigma \\ C\end{array}}{C}\ i_1 \dots i_5$$

where a is different from t, does not occur in F, G or any undischarged assumptions of Π except F^x_a and $\exists! a$; and b does not occur in F, G, C or any undischarged assumptions of Σ except F^x_b, G^x_b and $\exists! b$.

$$IE^{3p}: \quad \cfrac{Ix[F, \exists! x] \qquad \begin{array}{c}\underbrace{[F^x_a]^i, [\exists! a]^j}\\ \Pi \\ C\end{array}}{C}\ i,j \qquad IE^{5p}: \quad \cfrac{Ix[F, x = t] \qquad \exists! t \qquad \begin{array}{c}\underbrace{[F^x_a]^i, [\exists! a]^j}\\ \Pi \\ C\end{array}}{C}\ i,j$$

where a does not occur in F, C or any undischarged assumptions of Π except F^x_a and $\exists! a$.

$$IE^{2p}: \quad \cfrac{Ix[F, \exists! x] \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad F^x_{t_2} \qquad A^x_{t_1}}{A^x_{t_2}}$$ where A is an atomic formula.

$$IE^{4p}: \quad \cfrac{Ix[F, x = t_2] \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad A^x_{t_1}}{A^x_{t_2}}$$ where A is an atomic formula.

Sequent Calculus Rules for I for CPF

$$(RI) \quad \frac{\Gamma \Rightarrow \Delta, F^x_t \qquad \Gamma \Rightarrow \Delta, G^x_t \qquad \Gamma \Rightarrow \Delta, \exists! t \qquad \exists! a, F^x_a, \Gamma \Rightarrow \Delta, a = t}{\Gamma \Rightarrow \Delta, Ix[F, G]}$$

where a does not occur in the conclusion.

$$(LI^{1p}) \quad \frac{\Gamma \Rightarrow \Delta, F^x_t \qquad \Gamma \Rightarrow \Delta, \exists! t \qquad F^x_a, \exists! a, \Gamma \Rightarrow \Delta, a = t \qquad F^x_b, G^x_b, \exists! b, \Gamma \Rightarrow \Delta}{Ix[F, G], \Gamma \Rightarrow \Delta}$$

where a and b do not occur in the conclusion.

$$(LI^{2p}) \quad \frac{\Gamma \Rightarrow \Delta, F^x_{t_1} \qquad \Gamma \Rightarrow \Delta, F^x_{t_2} \qquad \Gamma \Rightarrow \Delta, \exists! t_1 \qquad \Gamma \Rightarrow \Delta, \exists! t_2 \qquad \Gamma \Rightarrow \Delta, A^x_{t_2}}{Ix[F, \exists! x], \Gamma \Rightarrow \Delta, A^x_{t_1}}$$

where A is an atomic formula.

$$(LI^{3p}) \quad \frac{F^x_a, \exists! a, \Gamma\Delta}{Ix[F, \exists! x], \Gamma\Delta}$$

where a does not occur in the conclusion.

$$(LI^{4p}) \quad \frac{\Gamma \Rightarrow \Delta, F^x_{t_1} \qquad \Gamma \Rightarrow \Delta, \exists! t_1 \qquad \Gamma \Rightarrow \Delta, \exists! t_2 \qquad \Gamma \Rightarrow \Delta, A^x_{t_2}}{Ix[F, x = t_2], \Gamma \Rightarrow \Delta, A^x_{t_1}}$$

where A is an atomic formula.

$$(LI^{5p}) \quad \frac{\Gamma \Rightarrow \Delta, \exists! t \qquad F^x_a, \exists! a, \Gamma \Rightarrow \Delta}{Ix[F, x = t], \Gamma \Rightarrow \Delta}$$

where a does not occur in the conclusion.

Assessing the Rules for I in IPF and CPF

Call the result of adding the new rules for I in natural deduction to IPF IPF^I and those of sequent calculus to CPF CPF^I.

Both systems have desirable proof theoretic properties: Deductions in IPF^I normalise, cut can be eliminated from those in CPF^I. The resulting theory of definite descriptions is also original.

But the rules for I are excessively complicated. The motivation was to stay close to standard positive free logic with a term forming ι operator by exploiting an equivalence that holds for formulas with ι terms to find rules for the binary quantifier I. Although it may be said to take to heart a tenet of Hintikka's and Lambert's, it is rather removed from standard systems.

If the aim is to stay close to standard systems, IPF^I and CPF^I are maybe a little disappointing; if the aim is to formulate a new theory of definite descriptions suitable I in positive free logic, there is a simpler way.

Note the Existence Assumptions in the Rule for I

$$II: \quad \frac{F^x_t \qquad G^x_t \qquad \exists! t \qquad \begin{matrix}[F^x_a]^i \ [\exists! a]^j \\ \Pi \\ a = t\end{matrix}}{Ix[F, G]}\ {\scriptstyle i,j}$$

where t is free for x in F and in G, and a does not occur in t nor in any undischarged assumptions in Π except F^x_a and $\exists! a$.

$$IE^1: \quad \frac{Ix[F, G] \qquad \begin{matrix}[F^x_a]^i \ [G^x_a]^j \ [\exists! a]^k \\ \Pi \\ C\end{matrix}}{C}\ {\scriptstyle i,j,k}$$

where a is not free in F, G, C nor any undischarged assumptions in Π except F^x_a, G^x_a and $\exists! a$.

$$IE^2: \quad \frac{Ix[F, G] \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad F^x_{t_2} \qquad A^x_{t_1}}{A^x_{t_2}}$$

where t_1 and t_2 are free for x in F and A is atomic.

Alternative Rules for $\imath$ for Positive Free Logic

What distinguishes positive from negative free logic is that truth or assertibility is not dependent on terms having referents: atomic sentences may be true even if some of their terms do not refer.

This applies to definite descriptions, too. When they are formalised by a binary quantifier in the context of complete sentences, the existence assumptions of the negative free logic should be relaxed or altogether given up. There not many options:

(1) Drop the existence assumptions of $\imath I$: then unique *existence* of an F is no longer required for 'The F is G' to be derivable, only uniqueness.
(2) Drop the discharged existence assumption in $\imath E^1$, which mirrors that of $\exists E$.
(3) $\imath E^2$ has two symmetric existence assumptions. If one goes, both go: dropping them also means that the identity of t_1 and t_2 no longer depends on their existence, but uniqueness still follows.

Natural Deduction Rules for I for Positive Free Logic

$$II: \quad \dfrac{F^x_t \qquad G^x_t \qquad \begin{array}{c}[F^x_a]^i \\ \Pi \\ a = t\end{array}}{Ix[F, G]}\, i$$

where a is different from t and does not occur in F, G or any undischarged assumptions of Π except F^x_a.

$$IE^1: \quad \dfrac{Ix[F, G] \qquad \begin{array}{c}\underbrace{[F^x_a]^i,\ [G^x_a]^j} \\ \Pi \\ C\end{array}}{C}\, i,j$$

where a is not free in C or any undischarged assumptions in Π except F^x_a, G^x_a.

$$IE^2: \quad \dfrac{Ix[F, G] \qquad F^x_{t_1} \qquad F^x_{t_2} \qquad C^x_{t_2}}{C^x_{t_1}}$$

where C is an atomic formula. (The general case follows by induction.)

Sequent Calculus Rules for I for Positive Free Logic

$$(RI) \quad \frac{\Gamma \Rightarrow \Delta, A^x_t \qquad \Gamma \Rightarrow \Delta, B^x_t \qquad A^x_a, \Gamma \Rightarrow \Delta, a = t}{\Gamma \Rightarrow \Delta, Ix[A, B]}$$

$$(LI^1) \quad \frac{A^x_a, B^x_a, \Gamma \Rightarrow \Delta}{Ix[A, B], \Gamma \Rightarrow \Delta}$$

$$(LI^2) \quad \frac{\Gamma \Rightarrow \Delta, A^x_{t_1} \qquad \Gamma \Rightarrow \Delta, A^x_{t_2} \qquad \Gamma \Rightarrow \Delta, C^x_{t_2}}{Ix[A, B], \Gamma \Rightarrow \Delta, C^x_{t_1}}$$

where in (RI) and (LI^1), a does not occur in the conclusion, and in (LI^2) C is an atomic formula. (The general case follows by induction.)

In Favour of the Revised Rules for I

The revised rules for I are simple and straightforward. They are motivated by considerations characteristic for positive free logic:

Negative free logic takes on Russell's analysis that the occurrence of a definite description 'the F' in a sentence indicates the existence and uniqueness of an F. To modify this to suit positive free logic there is really only one option: give up the requirement of existence.Thus we keep uniqueness, and this makes sense: with 'The F is G' we aim to speak of only one F, whether it exists or not. This is the effect of giving up the existence assumptions.

Added to CPF, resulting the system is cut free, added to IPF, the result permits normalisation.

But note that this goes against the views of some free logicians: Lambert thinks that 'The F is G' can be true not only if there is no F, but also if there is more than one.

Some Properties of I in Positive Free Logic

The derivability of a formula analogous to $\iota x(x = t) = t$ is immediate. Let F and G be $x = t$ in II:

$$\dfrac{t = t \qquad t = t \qquad [a = t]^1}{Ix[x = t, x = t]}\;{\scriptstyle 1}$$

This is desirable: It is an important principle for Lambert, but he has to add it as an axiom. Here it is for free.

An analogue of one half of Lambert's Law is derivable.

$$\dfrac{Ix[A, x = b] \qquad \dfrac{\dfrac{\dfrac{[a = b]^1 \qquad \dfrac{[c = b]^3 \qquad [A^x_c]^4}{A^x_b}}{A^x_a} \qquad \dfrac{Ix[A, x = b] \qquad [A^x_a]^2 \qquad \dfrac{[c = b]^3 \qquad [A^x_c]^4}{A^x_b}}{a = b}}{A^x_a \leftrightarrow a = b}\;{\scriptstyle 1,2}}{\forall x(A \leftrightarrow x = b)}}{\forall x(A \leftrightarrow x = b)}\;{\scriptstyle 3,4}$$

The other half is not derivable. The unique existence of an A is not sufficient for $Ix[A, x = b]$, because there might also be a non-existent A, in which case $Ix[A, x = b]$ is false.

Semantics. I

A *structure* $\mathfrak{A}$ is a function from the expressions of the language $\mathcal{L}$ of CPF^{I} to elements, a (possibly empty) subset, the sets of n-tuples of and operations on a non-empty set $|\mathfrak{A}|$, called the *domain of* $\mathfrak{A}$, such that:

1. $\mathfrak{A}$ assigns to the quantifier $\forall$ a (possibly empty) set $|\mathfrak{A}^{\forall}| \subseteq |\mathfrak{A}|$ called the *inner domain* or the *domain of quantification* of $\mathfrak{A}$.
2. $\mathfrak{A}$ assigns to the predicate $\exists!$ the set $|\mathfrak{A}^{\forall}|$.
3. $\mathfrak{A}$ assigns to each n-place predicate symbol P an n-ary relation $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^{n}$.
4. $\mathfrak{A}$ assigns to each constant symbol c an element $c^{\mathfrak{A}}$ of $|\mathfrak{A}|$.
5. $\mathfrak{A}$ assigns to each n-place function symbol f an n-ary operation $f^{\mathfrak{A}}$ on $|\mathfrak{A}|$, i.e. $f^{\mathfrak{A}}\colon |\mathfrak{A}|^{n} \to |\mathfrak{A}|$.

Semantics. II

Let s be a function from the variables for $\mathcal{L}$ to the domain of $|\mathfrak{A}|$. Then:

1. For each variable x, $\overline{s}(x) = s(x)$
2. For each constant symbol c, $\overline{s}(c) = c^{\mathfrak{A}}$.
3. For terms $t_1 \dots t_n$, n-place function symbols f, $\overline{s}(ft_1 \dots t_n) = f^{\mathfrak{A}}(\overline{s}(t_1) \dots \overline{s}(t_n))$

Satisfaction is defined explicitly for the atomic formulas of $\mathcal{L}$:

1. $\models_{\mathfrak{A}} t_1 = t_2\ [s]$ iff $\overline{s}(t_1) = \overline{s}(t_2)$.
2. $\models_{\mathfrak{A}} \exists! t\ [s]$ iff $\overline{s}(t) \in |\mathfrak{A}^{\forall}|$.
3. For n-place predicate parameters P, $\models_{\mathfrak{A}} Pt_1 \dots t_n\ [s]$ iff $\langle \overline{s}(t_1) \dots \overline{s}(t_n) \rangle \in P^{\mathfrak{A}}$.

And for the rest by recursion. $s(x|d)$ is like s, only that it assigns d to the variable x:

1. For atomic formulas, as above.
2. $\models_{\mathfrak{A}} \neg A\ [s]$ iff $\not\models_{\mathfrak{A}} A\ [s]$.
3. $\models_{\mathfrak{A}} A \rightarrow B\ [s]$ iff either $\not\models_{\mathfrak{A}} A\ [s]$ or $\models_{\mathfrak{A}} B\ [s]$.
4. $\models_{\mathfrak{A}} \forall x A\ [s]$ iff for every $d \in |\mathfrak{A}^{\forall}|$, $\models_{\mathfrak{A}} A\ [s(x|d)]$.
5. $\models_{\mathfrak{A}} Ix[A, B]\ [s]$ iff there is $d \in |\mathfrak{A}|$ such that: $\models_{\mathfrak{A}} A\ [s(x|d)]$, there is no other $e \in |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} A\ [s(x|e)]$, and $\models_{\mathfrak{A}} B\ [s(x|d)]$.

$\models_{\mathfrak{A}} Ix[F, G]\ [s]$ iff there is exactly one element in the domain of $\mathfrak{A}$ such that $\mathfrak{A}$ satisfies A with s modified to assign that element to x, and $\mathfrak{A}$ satisfies B with the same modified s.

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