AXIOMS AS RULES:

HOW TO REASON WITH MATHEMATICAL THEORIES

UCL Computer Science

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EXTENDD SEMINAR

AXIOMS

- FUNDAMENTAL STATEMENT ACCEPTED AS TRUE
- FOUNDATION FOR FURTHER REASONING

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AXIOMS

logical   non-logical

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- FUNDAMENTAL STATEMENT ACCEPTED AS TRUE
- FOUNDATION FOR FURTHER REASONING

AXIOMS

logical

A implies (B implies A)

non-logical

$\forall a, b.\ a+b = b+a$

AXIOMS

- FUNDAMENTAL STATEMENT ACCEPTED AS TRUE
- FOUNDATION FOR FURTHER REASONING

Axioms

logical

A implies (B implies A)

non-logical

$\forall a, b.\ a+b = b+a$

↪ MATHEMATICAL THEORIES

THE AXIOMS-AS-RULES PROBLEM

PROJECTIVE GEOMETRY [Negri & von Plato] (UNIQUENESS)

$$\forall a,b,l,m.\ a \in l \wedge a \in m \wedge b \in l \wedge b \in m \supset a = b \vee l = m.$$

THE AXIOMS-AS-RULES PROBLEM

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$$\forall a,b,l,m.\ a\in l \wedge a\in m \wedge b\in l \wedge b\in m \supset a=b \vee l=m.$$

$$\frac{\Gamma \vdash \Delta, a\in l \quad \Gamma \vdash \Delta, a\in m \quad \Gamma \vdash \Delta, b\in l \quad \Gamma \vdash \Delta, b\in m}{\Gamma \vdash \Delta, a=b, l=m}\ Uni_R$$

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$$\forall a,b,l,m.\ a \in l \wedge a \in m \wedge b \in l \wedge b \in m \supset a = b \vee l = m.$$

$$\frac{\Gamma \vdash \Delta, a \in l \qquad \Gamma \vdash \Delta, a \in m \qquad \Gamma \vdash \Delta, b \in l \qquad \Gamma \vdash \Delta, b \in m}{\Gamma \vdash \Delta, a = b, l = m}\ Uni_n$$

$$\frac{\Gamma, a = b \vdash \Delta \qquad \Gamma, l = m \vdash \Delta}{\Gamma, a \in l, a \in m, b \in l, b \in m \vdash \Delta}\ Uni_p$$

THE AXIOMS-AS-RULES PROBLEM

EUCLIDEAN GEOMETRY (CONGRUENCE)

$\forall x, y.\ \mathrm{cong}\ x y\ y x.$

$\forall x, y, z, w, r, s.\ \mathrm{cong}\ x y\ z w \wedge \mathrm{cong}\ x y\ r s \supset \mathrm{cong}\ z w\ r s.$

THE AXIOMS-AS-RULES PROBLEM

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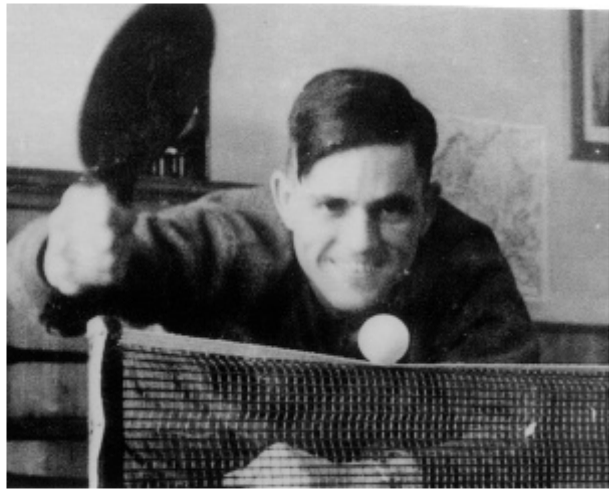
$$\frac{\Gamma \vdash \mathrm{cong}\ x y\ z w, \Delta \qquad \Gamma \vdash \mathrm{cong}\ x y\ r s}{\Gamma \vdash \mathrm{cong}\ z w\ r s}\ \mathrm{cong}_n$$

$$\frac{\Gamma, \mathrm{cong}\ z w\ r s \vdash \Delta}{\Gamma, \mathrm{cong}\ x y\ z w, \mathrm{cong}\ x y\ r s \vdash \Delta}\ \mathrm{cong}_p$$

THE AXIOMS-AS-RULES PROBLEM

ARITHMETIC WITHOUT COMPLETE INDUCTION [Gentzen]

Equality:	
$\forall x\,(x = x)$	(reflexivity)
$\forall x\,\forall y\,(x = y \supset y = x)$	(symmetry)
$\forall x\,\forall y\,\forall z\,((x = y \,\&\, y = z) \supset x = z)$	(transitivity)
One:	
$\exists x\,(\mathrm{One}\ x)$	(existence of 1)
$\forall x\,\forall y\,((\mathrm{One}\ x \,\&\, \mathrm{One}\ y) \supset x = y)$	(uniqueness of 1)
Predecessor:	
$\forall x\,\exists y\,(x\mathrm{Pr}y)$	(existence of successor)
$\forall x\,\forall y\,(x\mathrm{Pr}y \supset \neg\,\mathrm{One}\ y)$	(1 has no predecessor)
$\forall x\,\forall y\,\forall z\,\forall u\,((x\mathrm{Pr}y \,\&\, z\mathrm{Pr}u \,\&\, x = z) \supset y = u)$	(uniqueness of successor)
$\forall x\,\forall y\,\forall z\,\forall u\,((x\mathrm{Pr}y \,\&\, z\mathrm{Pr}u \,\&\, y = u) \supset x = u)$	(uniqueness of predecessor).



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THE AXIOMS-AS-RULES PROBLEM

ARITHMETIC WITHOUT COMPLETE INDUCTION [Gentzen]

Equality:

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$\forall x\,\forall y\,((\mathrm{One}\;x\;\&\;\mathrm{One}\;y) \supset x = y)$	(uniqueness of 1)

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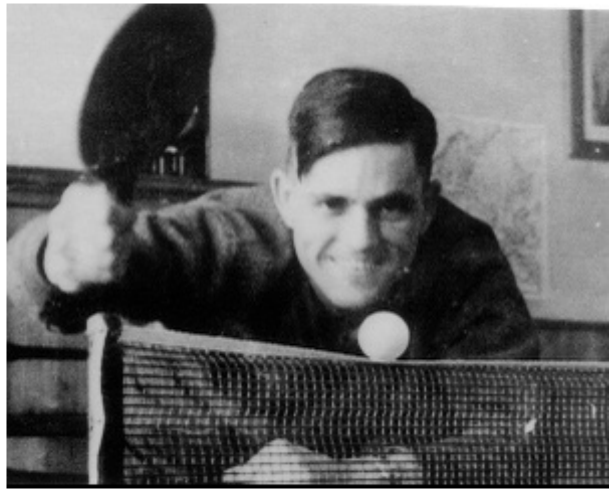

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A formula $\mathfrak{B}$ is called *derivable* in arithmetic without complete induction, if there is an *LK*-derivation for a sequent

$$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \rightarrow \mathfrak{B}$$

in which $\mathfrak{A}_1, \ldots, \mathfrak{A}\mu$ are axiom formulae of arithmetic.

HOW ?

METHOD I: ADD NON LOGICAL AXIOMS

ASSUME:

$\vdash P \supset Q$ $\qquad\qquad$ $\vdash P$

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ASSUME:

$$\vdash P \supset Q \qquad\qquad \vdash P$$

THEN:

$$\dfrac{\dfrac{}{\vdash P} \qquad \dfrac{\dfrac{}{\vdash P \supset Q} \qquad \dfrac{\dfrac{}{P \vdash P} \quad \dfrac{}{Q \vdash Q}}{P, P \supset Q \vdash Q}\,{\supset}L}{P \vdash Q}\,\mathrm{cut}}{\vdash Q}\,\mathrm{cut}$$

GIRARD: THE HAUPTSATZ FAILS!

METHOD II: ADD MATHEMATICAL BASIC SEQUENTS

ASSUME:

$P \vdash Q \qquad \vdash P$

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ASSUME:

$$P \vdash Q \qquad\qquad \vdash P$$

THEN:

$$\dfrac{\overline{\vdash P} \qquad \overline{P \vdash Q}}{\vdash Q}\,\text{cut}$$

METHOD II: ADD MATHEMATICAL BASIC SEQUENTS

ASSUME:

$$P \vdash Q \qquad\qquad \vdash P$$

THEN:

$$\dfrac{\dfrac{}{\vdash P} \qquad \dfrac{}{P \vdash Q}}{\vdash Q}\ \text{cut}$$

GIRARD: HAUPTSATZ DOES NOT EXTEND TO BASIC SEQUENTS!

METHOD III: ADD AXIOMS AS THEORIES

$$\dfrac{\dfrac{}{P \vdash P} \qquad \dfrac{}{Q \vdash Q}}{P, P \supset Q \vdash Q}$$

METHOD III: ADD AXIOMS AS THEORIES

$$\dfrac{\dfrac{}{P \vdash P} \qquad \dfrac{}{Q \vdash Q}}{P, P \supset Q \vdash Q}$$

GENTZEN :

3.21. A 'contradiction' $\mathfrak{A} \,\&\, \neg\, \mathfrak{A}$ is derivable in our system if and only if there exists an LK-derivation for a sequent with an *empty succedent* and with arithmetic axiom formulae in the antecedent, viz.:

From $\Gamma \rightarrow \mathfrak{A} \,\&\, \neg\, \mathfrak{A}$ we obtain $\Gamma \rightarrow$ in the following way:

$$\dfrac{\Gamma \rightarrow \mathfrak{A} \,\&\, \neg\, \mathfrak{A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{A} \rightarrow \mathfrak{A}}{\neg\, \mathfrak{A}, \mathfrak{A} \rightarrow}\ \neg\text{-}IA}{\mathfrak{A} \,\&\, \neg\, \mathfrak{A}\ \mathfrak{A} \rightarrow}\ \&\text{-}IA}{\mathfrak{A}, \mathfrak{A} \,\&\, \neg\, \mathfrak{A} \rightarrow}\ \text{interchange}}{\mathfrak{A} \,\&\, \neg\, \mathfrak{A}, \mathfrak{A} \,\&\, \neg\, \mathfrak{A} \rightarrow}\ \&\text{-}IA}{\mathfrak{A} \,\&\, \neg\, \mathfrak{A} \rightarrow}\ \text{contraction}}{\Gamma \rightarrow}\ \text{cut.}$$

The converse is obtained by carrying out a thinning in the succedent.

Thus, if our arithmetic is inconsistent, there exists an LK-derivation with the endsequent

$$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \rightarrow,$$

where $\mathfrak{A}_1, \ldots, \mathfrak{A}\mu$ are arithmetic axiom formulae.

3.22. We now apply the *sharpened Hauptsatz* (2.1). The arithmetic axiom

METHOD IV : ADD NON-LOGICAL INFERENCE RULES

$$\frac{\Gamma, Q \vdash C}{\Gamma, P \vdash C}\; P \supset Q \qquad\qquad \frac{\Gamma, P \vdash C}{\Gamma \vdash C}\; P$$

[Negri & von Plato, Simpson, Viganò]

← SARA

METHOD IV: ADD NON-LOGICAL INFERENCE RULES

$$\frac{\Gamma, Q \vdash C}{\Gamma, P \vdash C}\; P \supset Q \qquad\qquad \frac{\Gamma, P \vdash C}{\Gamma \vdash C}\; P$$

[Negri & von Plato, Simpson, Viganò]

$$\dfrac{\dfrac{Q \vdash Q}{P \vdash Q}\; P \supset Q}{\vdash Q}\; P$$

HOW ?

AXIOMS-AS-RULES

OBJECT	FIRST-ORDER LOGIC	MATHEMATICAL THEORY
	↓	↓
REASONING	SEQUENT CALCULUS	INFERENCE RULES

AXIOMS-AS-RULES

OBJECT	FIRST-ORDER LOGIC	MATHEMATICAL THEORY
	↓	↓
REASONING	SEQUENT CALCULUS	INFERENCE RULES

BIPOLARS + FOCUSING

=

SYNTHETIC INFERENCE RULES

AXIOMS-AS-RULES

OBJECT	FIRST-ORDER LOGIC	MATHEMATICAL THEORY
	↓	↓
REASONING	SEQUENT CALCULUS	INFERENCE RULES

SYSTEMATICALLY

UNIFORMLY

BIPOLARS + FOCUSING

=

SYNTHETIC INFERENCE RULES

AXIOMS-AS-RULES

OBJECT	FIRST-ORDER LOGIC	MATHEMATICAL THEORY
	↓	↓
REASONING	SEQUENT CALCULUS	INFERENCE RULES

GENERALIZATION

IMPLEMENTATION

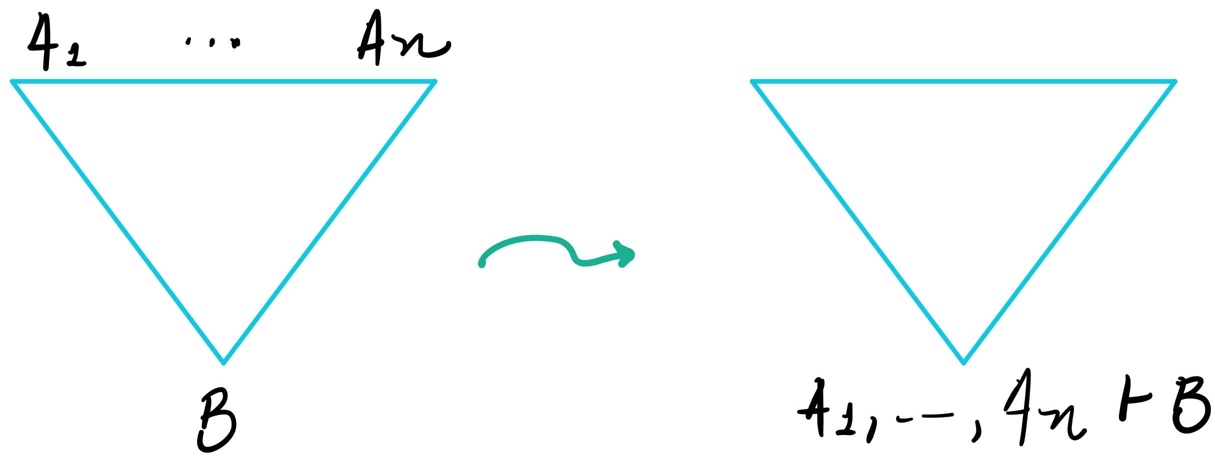
CUT-ELIMINATION

BIPOLARS + FOCUSING

=

SYNTHETIC INFERENCE RULES

REASONING: SEQUENT CALCULUS



$\Gamma = A_1, \ldots, A_n$ CONTEXT

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset_R \qquad \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \supset B \vdash C}\ \supset_L$$

CUT-ELIMINATION: $$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

POLARIZATION [Joinet et al]

$$\frac{\Gamma \vdash A_1 \qquad \Gamma, A_0 \vdash C}{\Gamma, A_1 \supset A_0 \vdash C} \supset_L$$

JEAN-BAPTISTE

POLARIZATION [Joinet et al]

$$\frac{\Gamma \vdash A_1 \qquad \overline{\Gamma, A_0 \vdash C}}{\Gamma, A_1 \supset A_0 \vdash C} \supset_L$$

NEGATIVE PROTOCOL

POLARIZATION [Joinet et al]

$$\dfrac{\Gamma \vdash A_1 \qquad \Gamma, A_0 \vdash C}{\Gamma, A_1 \supset A_0 \vdash C}\supset_L$$

NEGATIVE PROTOCOL

$$\dfrac{\Gamma \vdash A_1 \qquad \dfrac{\Gamma \vdash A_2 \qquad \dfrac{\Gamma \vdash A_3 \qquad \dfrac{\Gamma \vdash A_4 \qquad \dfrac{B = A_0}{\Gamma, A_0 \vdash B}}{\Gamma, A_4 \supset A_0 \vdash B}\supset_L}{\Gamma, A_3 \supset A_4 \supset A_0 \vdash B}\supset_L}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\supset_L}{\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\supset_L$$

BACK-CHAINING!

POLARIZATION [Joinet et al]

$$\frac{\Gamma \vdash A_1 \qquad \Gamma, A_0 \vdash C}{\Gamma, A_1 \supset A_0 \vdash C} \supset_L$$

POSITIVE PROTOCOL

POLARIZATION [Joinet et al]

$$\frac{\Gamma \vdash A_1 \qquad \Gamma, A_0 \vdash C}{\Gamma, A_1 \supset A_0 \vdash C} \supset_L$$

POSITIVE PROTOCOL

$$\dfrac{\dfrac{A_1 \in \Gamma}{\Gamma \vdash A_1} \qquad \dfrac{\dfrac{A_2 \in \Gamma}{\Gamma \vdash A_2} \qquad \dfrac{\dfrac{A_3 \in \Gamma}{\Gamma \vdash A_3} \qquad \dfrac{\dfrac{A_4 \in \Gamma}{\Gamma \vdash A_4} \qquad A_0 \vdash B}{\Gamma, A_4 \supset A_0 \vdash B}}{\Gamma, A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}$$

FORWARD-CHAINING

FIBONACCI !

$$\Delta = \{ fib(0,0), fib(1,1), \forall n, x, y. [fib(n,x) \wedge fib(n+1,y) \supset fib(n+2, x+y)] \}$$

$fib(n,N) \rightsquigarrow N$ is the nth Fibonacci number

FIBONACCI !

$$\Delta = \{ \mathrm{fib}(0,0), \mathrm{fib}(1,1), \forall n,x,y. [\mathrm{fib}(n,x) \wedge \mathrm{fib}(n+1,y) \supset \mathrm{fib}(n+2, x+y)] \}$$

Negative protocol:

$$\dfrac{\overline{\Delta \vdash \mathrm{fib}(0,0)} \qquad \dfrac{\overline{\Delta \vdash \mathrm{fib}(1,1)} \qquad \dfrac{\begin{array}{c} \mathrm{fib}(0,0) \\ \mathrm{fib}(1,1) \\ \nabla \\ \Delta \vdash \mathrm{fib}(2,1) \end{array} \qquad \dfrac{\begin{array}{c} \mathrm{fib}(1,1) \\ \mathrm{fib}(2,1) \\ \nabla \\ \Delta \vdash \mathrm{fib}(3,2) \end{array} \qquad \overline{\Delta, \mathrm{fib}(4,3) \vdash \mathrm{fib}(4,3)}}{\Delta, \mathrm{fib}(3,2) \supset \mathrm{fib}(4,3) \vdash \mathrm{fib}(4,3)}}{\Delta, \mathrm{fib}(2,1) \supset \mathrm{fib}(3,2) \supset \mathrm{fib}(4,3) \vdash \mathrm{fib}(4,3)}}{\Delta, \mathrm{fib}(1,1) \supset \mathrm{fib}(2,1) \supset \mathrm{fib}(3,2) \supset \mathrm{fib}(4,3) \vdash \mathrm{fib}(4,3)}}{\Delta, \mathrm{fib}(0,0) \supset \mathrm{fib}(1,1) \supset \mathrm{fib}(2,1) \supset \mathrm{fib}(3,2) \supset \mathrm{fib}(4,3) \vdash \mathrm{fib}(4,3)} \; L\supset$$

FIBONACCI!

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$$\dfrac{\dfrac{}{\Delta \vdash \mathsf{fib}(0,0)} \quad \dfrac{\dfrac{}{\Delta \vdash \mathsf{fib}(1,1)} \quad \dfrac{\begin{matrix}\overset{\substack{\mathsf{fib}(0,0)\\ \mathsf{fib}(1,1)}}{\nabla}\\ \Delta \vdash \mathsf{fib}(2,1)\end{matrix} \quad \dfrac{\begin{matrix}\overset{\substack{\mathsf{fib}(1,1)\\ \mathsf{fib}(2,1)}}{\nabla}\\ \Delta \vdash \mathsf{fib}(3,2)\end{matrix} \quad \dfrac{}{\Delta, \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta, \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta, \mathsf{fib}(2,1) \supset \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta, \mathsf{fib}(1,1) \supset \mathsf{fib}(2,1) \supset \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta, \mathsf{fib}(0,0) \supset \mathsf{fib}(1,1) \supset \mathsf{fib}(2,1) \supset \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}\ L\supset$$

UNIQUE PROOF - EXPONENTIAL IN SIZE!

FiBONACCi !

$\Delta = \{ fib(0,0), fib(1,1), \forall n, x, y . [fib(n,x) \wedge fib(n+1,y)$

$\supset fib(n+2, x+y)] \}$

Positive protocol:

$$
\dfrac{\dfrac{}{\Delta \vdash \mathsf{fib}(0,0)} \qquad \dfrac{\dfrac{}{\Delta \vdash \mathsf{fib}(1,1)} \qquad \dfrac{\dfrac{\dfrac{}{\Delta' \vdash \mathsf{fib}(2,1)} \qquad \dfrac{\dfrac{\dfrac{}{\Delta'' \vdash \mathsf{fib}(3,2)} \qquad \dfrac{}{\Delta'', \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta'', \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta', \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta, \mathsf{fib}(2,1) \supset \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta, \mathsf{fib}(2,1) \supset \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta, \mathsf{fib}(1,1) \supset \mathsf{fib}(2,1) \supset \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)}}{\Delta, \mathsf{fib}(0,0) \supset \mathsf{fib}(1,1) \supset \mathsf{fib}(2,1) \supset \mathsf{fib}(3,2) \supset \mathsf{fib}(4,3) \vdash \mathsf{fib}(4,3)} \; L\supset
$$

where $\Delta' = \Delta, \mathsf{fib}(2,1)$ and $\Delta'' = \Delta', \mathsf{fib}(3,2)$.

FIBONACCI!

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where $\Delta' = \Delta, \mathsf{fib}(2,1)$ and $\Delta'' = \Delta', \mathsf{fib}(3,2)$.

MANY PROOFS – THE SMALLEST IS LINEAR IN SIZE!

POLARIZING CONNECTIVES

CLASSICAL:

$$\wedge^-, \wedge^+, t^-, t^+, \vee^-, \vee^+, f^-, f^+$$

$\forall$ negative $\qquad\qquad$ $\exists$ positive

INTUITIONISTIC:

$$\wedge^-, \wedge^+, t^-, t^+, \quad \vee^+, \quad f^+$$

$\forall$ negative $\qquad\qquad$ $\exists$ positive

$\supset$ negative

POLARIZING FORMULAS

$$(P_1 \supset P_2) \vee (Q_1 \supset Q_2)$$

POLARIZING FORMULAS

$$(P_1 \supset P_2) \vee^{+} (Q_1 \supset Q_2)$$

FORMULA HIERARCHY

$$\mathcal{P}_0$$

$$\updownarrow$$

$$\mathcal{N}_0$$

P, N

FORMULA HIERARCHY

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[d, leftrightarrow] \arrow[dr] & \\
\mathcal{N}_0 \arrow[r] & \mathcal{N}_1
\end{tikzcd}$$

$$p_1 \supset p_2$$

FORMULA HIERARCHY

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[dr] & \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] & \mathcal{N}_2
\end{tikzcd}
$$

$$(\mathcal{N}_1 \vee^{+} \mathcal{N}_2) \wedge^{-} \mathcal{P}_1$$

FORMULA HIERARCHY

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 & \cdots
\end{tikzcd}$$

AGATA

BASED ON THE ORIGINAL IDEA OF [Ciabattoni et al.]

FORMULA HIERARCHY

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[dr] & \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] & \mathcal{N}_2
\end{tikzcd}
$$

$\mathcal{N}_2$ = BIPOLAR

FORMULA HIERARCHY

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[dr] & \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] & \mathcal{N}_2
\end{tikzcd}
$$

$\mathcal{N}_2$ = BIPOLAR

$$(P_1 \supset P_2) \vee^{+} (Q_1 \supset Q_2)$$

NOT BIPOLAR !

FORMULA HIERARCHY

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[dr] & \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] & \mathcal{N}_2
\end{tikzcd}$$

$\mathcal{N}_2$ = BIPOLAR

$$(P_1 \supset P_2) \vee^- (Q_1 \supset Q_2)$$

BIPOLAR !

FORMULA HIERARCHY

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[dr] & \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] & \mathcal{N}_2
\end{tikzcd}
$$

$\mathcal{N}_2$ = BIPOLAR

$$(P_1 \supset P_2) \vee^- (Q_1 \supset Q_2)$$

ONLY CLASSICAL $\rightsquigarrow$ BIPOLAR !

FOCUSING [Andreoli]

HOW TO PROVE

$$\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B$$

?

FOCUSING [Andreoli]

$$
\dfrac{\Gamma \vdash A_1 \qquad \dfrac{\Gamma \vdash A_2 \qquad \dfrac{\Gamma \vdash A_3 \qquad \dfrac{\Gamma \vdash A_4 \qquad \Gamma, A_0 \vdash B}{\Gamma, A_4 \supset A_0 \vdash B}}{\Gamma, A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma,\ A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}
$$

FOCUSING [Andreoli]

$$
\dfrac{\Gamma \vdash A_1 \Downarrow \qquad \dfrac{\Gamma \vdash A_2 \Downarrow \qquad \dfrac{\Gamma \vdash A_3 \Downarrow \qquad \dfrac{\Gamma \vdash A_4 \Downarrow \qquad \Gamma \Downarrow A_0 \vdash B}{\Gamma \Downarrow A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}
$$

FOCUSING [Andreoli]

Focus is lost if the right formula is negative

$$\dfrac{\Gamma \vdash A_1 \Downarrow \qquad \dfrac{\Gamma \vdash A_2 \Downarrow \qquad \dfrac{\Gamma \vdash A_3 \Downarrow \qquad \dfrac{\dfrac{\Gamma \vdash A_4 \Uparrow}{\Gamma \vdash A_4 \Downarrow} \qquad \Gamma \Downarrow A_0 \vdash B}{\Gamma \Downarrow A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}$$

FOCUSING [Andreoli]

Focus is lost if the left formula is positive

$$\dfrac{\Gamma \vdash A_1 \Downarrow \qquad \dfrac{\Gamma \vdash A_2 \Downarrow \qquad \dfrac{\Gamma \vdash A_3 \Downarrow \qquad \dfrac{\Gamma \vdash A_4 \Downarrow \qquad \dfrac{\Gamma \Uparrow A_0 \vdash B}{\Gamma \Downarrow A_0 \vdash B}}{\Gamma \Downarrow A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}$$

FOCUSING [Andreoli]

$$\cfrac{\Gamma \vdash A_1 \Downarrow \qquad \cfrac{\Gamma \vdash A_2 \Downarrow \qquad \cfrac{\Gamma \vdash A_3 \Downarrow \qquad \cfrac{\Gamma \vdash A_4 \Downarrow \qquad \Gamma \Downarrow A_0 \vdash B}{\Gamma \Downarrow A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma \Downarrow A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}$$

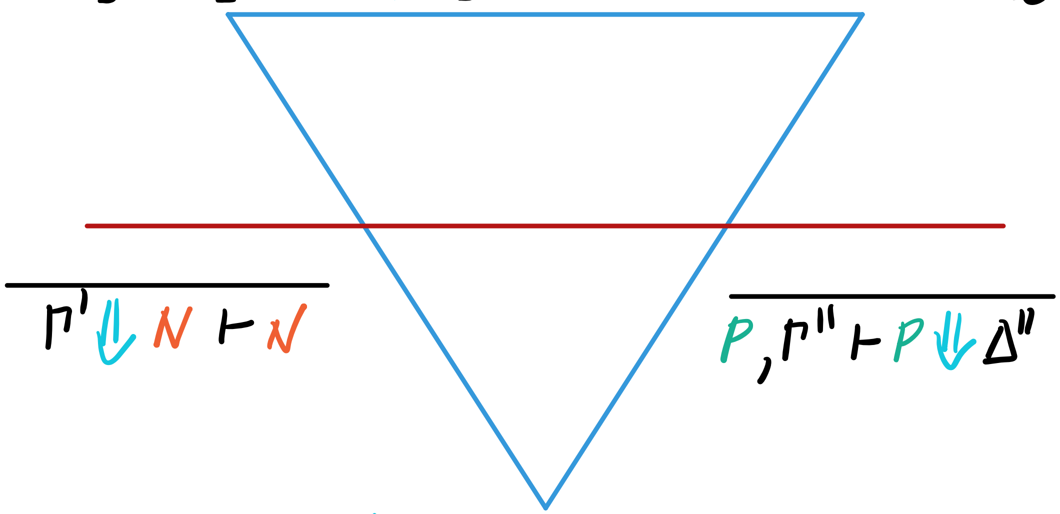
COMPLETE PROCEDURE

THE MAIN RESULT

ATOMS ATOMS

$$\Gamma_1 \Uparrow \overline{P}_1 \vdash \overline{Q}_1 \Uparrow \Delta_1 \quad \ldots \quad \Gamma_n \Uparrow \overline{P}_n \vdash \overline{Q}_n \Uparrow \Delta_n$$

NEGATIVE PHASE

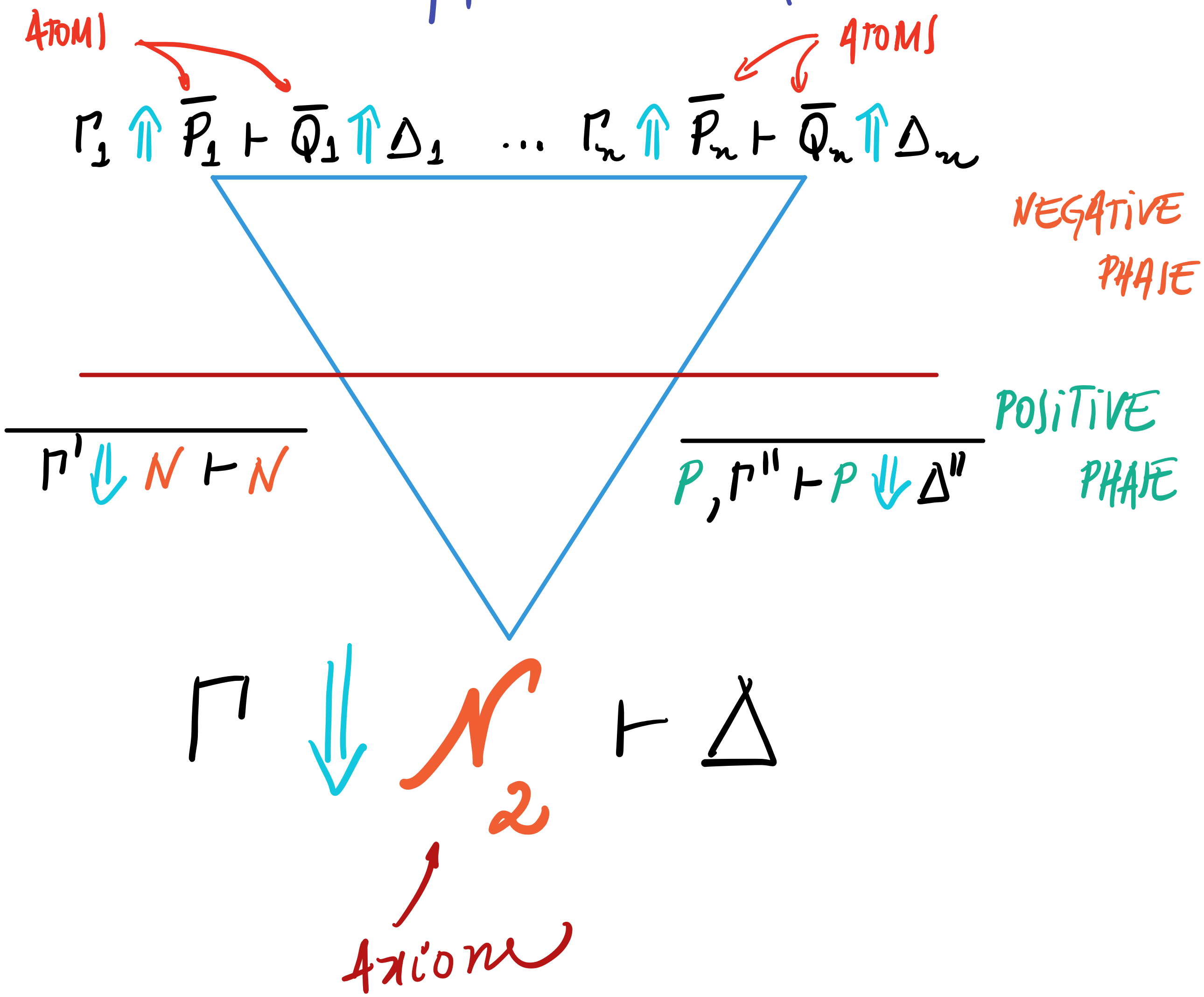


POSITIVE PHASE

$$\frac{}{\Gamma' \Downarrow N \vdash N} \qquad \frac{}{P, \Gamma'' \vdash P \Downarrow \Delta''}$$

$$\Gamma \Downarrow \mathcal{N}_2 \vdash \Delta$$

THE MAIN IDEA

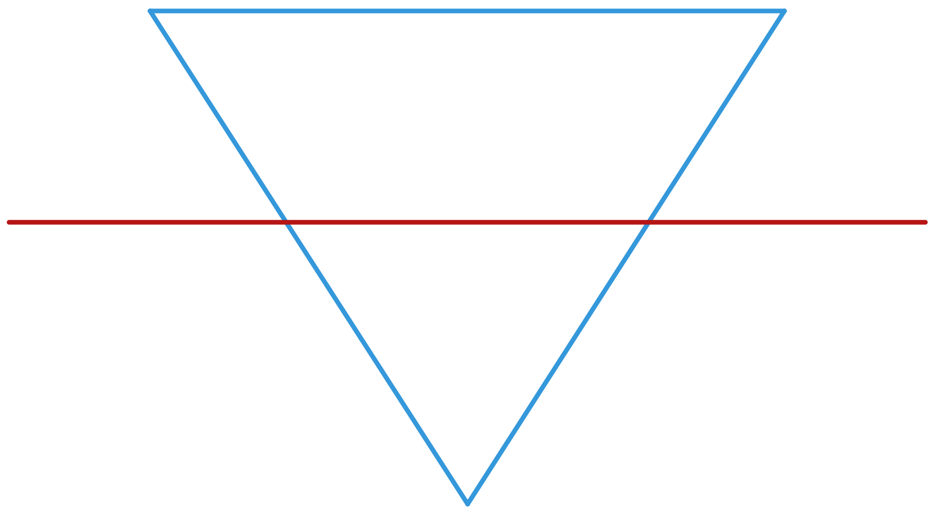


THE MAIN IDEA

ATOMS

ATOMS

$$\Gamma_1 \Uparrow \overline{P_1} \vdash \overline{Q_1} \Uparrow \Delta_1 \quad \dots \quad \Gamma_n \Uparrow \overline{P_n} \vdash \overline{Q_n} \Uparrow \Delta_n$$



NEGATIVE PHASE

$$\Gamma, \overline{P} \vdash \Delta, \overline{N}$$

THE MAIN IDEA

$$\frac{\Gamma_1, \overline{P}_1 \vdash \overline{Q}_1, \Delta_1 \quad \ldots \quad \Gamma_n, \overline{P}_n \vdash \overline{Q}_n, \Delta_n}{\Gamma, \overline{P} \vdash \Delta, \overline{N}}$$

IN ACTION!

$$\forall u, v, w. \left[(w \in u \supset w \in v) \supset u \subseteq v \right]$$

$$\Gamma \Downarrow \forall u, v, w. ((\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v) \vdash C$$

IN ACTION!

$$\forall u, v, w. [(w \in u \supset w \in v) \supset u \subseteq v]$$

$$\frac{\Gamma \Downarrow (\text{in } w\, u \supset \text{in } w\, v) \supset \text{subset } u\, v \vdash C}{\Gamma \Downarrow \forall u, v, w. ((\text{in } w\, u \supset \text{in } w\, v) \supset \text{subset } u\, v) \vdash C}\ \forall_L$$

IN ACTION!

$$\forall u, v, w.\, [(w \in u \supset w \in v) \supset u \subseteq v]$$

$$\dfrac{\dfrac{\Gamma \Downarrow \mathit{subset}\ u\ v \vdash C \qquad \Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Downarrow}{\Gamma \Downarrow (\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v \vdash C}\ {\supset_L}}{\Gamma \Downarrow \forall u, v, w.\, ((\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v) \vdash C}$$

IN ACTION!

$$\forall u, v, w. [(w \in u \supset w \in v) \supset u \subseteq v]$$

$$\cfrac{\Gamma \Downarrow \mathit{subset}\ u\ v \vdash C \qquad \cfrac{\Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Uparrow}{\Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Downarrow}\ \Uparrow}{\cfrac{\Gamma \Downarrow (\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v \vdash C}{\Gamma \Downarrow \forall u, v, w. ((\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v) \vdash C}}$$

IN ACTION!

$$\forall u, v, w.\,[(w \in u \supset w \in v) \supset u \subseteq v]$$

$$\dfrac{\Gamma \Downarrow \mathit{subset}\ u\ v \vdash c \qquad \dfrac{\dfrac{\Gamma, \mathit{in}\ w\ u \vdash \mathit{in}\ w\ v \Uparrow}{\Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Uparrow}\ \supset_R}{\Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Downarrow}}{\dfrac{\Gamma \Downarrow (\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v \vdash c}{\Gamma \Downarrow \forall u, v, w.\,((\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v) \vdash c}}$$

IN ACTION!

$$\forall u, v, w. [(w \in u \supset w \in v) \supset u \subseteq v]$$

subset negative!

$$\dfrac{\dfrac{\dfrac{}{\Gamma \Downarrow \mathit{subset}\ u\ v \vdash c} \qquad \dfrac{\dfrac{\Gamma, \mathit{in}\ w\ u \vdash \mathit{in}\ w\ v \Uparrow}{\Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Uparrow}}{\Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Downarrow}}{\Gamma \Downarrow (\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v \vdash c}}{\Gamma \Downarrow \forall u, v, w. ((\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v) \vdash c}$$

IN ACTION!

$$\forall u, v, w. \left[(w \in u \supset w \in v) \supset u \subseteq v \right]$$

subset negative!

$$
\dfrac{\dfrac{}{\Gamma \Downarrow \text{subset}\ u\ v \vdash \text{subset}\ u\ v} \qquad \dfrac{\dfrac{\Gamma, \text{in}\ w\ u \vdash \text{in}\ w\ v \Uparrow}{\Gamma \vdash \text{in}\ w\ u \supset \text{in}\ w\ v \Uparrow}}{\Gamma \vdash \text{in}\ w\ u \supset \text{in}\ w\ v \Downarrow}}{\dfrac{\Gamma \Downarrow (\text{in}\ w\ u \supset \text{in}\ w\ v) \supset \text{subset}\ u\ v \vdash \text{subset}\ u\ v}{\Gamma \Downarrow \forall u, v, w. ((\text{in}\ w\ u \supset \text{in}\ w\ v) \supset \text{subset}\ u\ v) \vdash \text{subset}\ u\ v}}
$$

IN ACTION!

$\forall u, v, w. [(w \in u \supset w \in v) \supset u \subseteq v]$

subset negative!

$$\frac{\Gamma, \textit{in } w\, u \vdash \textit{in } w\, v}{\Gamma \vdash \textit{subset } u\, v}$$

IN ACTION!

$$\forall u, v, w . \left[(w \in u \supset w \in v) \supset u \subseteq v \right]$$

subset negative!

$$\frac{\Gamma, \mathit{in}\ w\ u \vdash \mathit{in}\ w\ v}{\Gamma \vdash \mathit{subset}\ u\ v} \subseteq_n$$

IN ACTION!

$$\forall u, v, w.\,[(w \in u \supset w \in v) \supset u \subseteq v]$$

subset positive!

$$\dfrac{\dfrac{\Gamma \Uparrow \mathit{subset}\ u\ v \vdash c}{\Gamma \Downarrow \mathit{subset}\ u\ v \vdash c} \qquad \dfrac{\dfrac{\Gamma, \mathit{in}\ w\ u \vdash \mathit{in}\ w\ v \Uparrow}{\Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Uparrow}}{\Gamma \vdash \mathit{in}\ w\ u \supset \mathit{in}\ w\ v \Downarrow}}{\dfrac{\Gamma \Downarrow (\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v \vdash c}{\Gamma \Downarrow \forall u, v, w.\,((\mathit{in}\ w\ u \supset \mathit{in}\ w\ v) \supset \mathit{subset}\ u\ v) \vdash c}}$$

IN ACTION!

$\forall u, v, w . [(w \in u \supset w \in v) \supset u \subseteq v]$

subset positive!

$$\dfrac{\Gamma, \text{subset } u\ v \vdash c \qquad \Gamma, \text{in } w\ u \vdash \text{in } w\ v}{\Gamma \qquad \vdash c}$$

IN ACTION!

$$\forall u, v, w. [(w \in u \supset w \in v) \supset u \subseteq v]$$

subset positive!

$$\frac{\Gamma, \mathit{subset}\ u\ v \vdash c \qquad \Gamma, \mathit{in}\ w\ u \vdash \mathit{in}\ w\ v}{\Gamma \vdash c} \subseteq p$$

IN ACTION!

$\forall u, v, w . [(w \in u \supset w \in v) \supset u \subseteq v]$

$$\frac{\Gamma, \text{in } w\, u \vdash \text{in } w\, v}{\Gamma \vdash \text{subset } u\, v} \subseteq_n$$

$$\frac{\Gamma, \text{subset } u\, v \vdash c \qquad \Gamma, \text{in } w\, u \vdash \text{in } w\, v}{\Gamma \vdash c} \subseteq_p$$

IN GENERAL!

HORN CLAUSES:

$$\forall \bar{z}. (P_1 \wedge P_2 \wedge \ldots \wedge P_n \supset Q)$$

IN GENERAL!

HORN CLAUSES:

$$\forall \bar{z}. \left(P_1 \overset{+}{\wedge} P_2 \overset{+}{\wedge} \dots \overset{+}{\wedge} P_n \supset Q \right)$$

IN GENERAL!

HORN CLAUSES:

$$\forall \bar{z}. \left(P_1 \wedge^{+} P_2 \wedge^{+} \dots \wedge^{+} P_n \supset Q \right)$$

IN GENERAL!

HORN CLAUSES:

$$\forall \bar{z}. (P_1 \wedge^+ P_2 \wedge^+ \dots \wedge^+ P_n \supset Q)$$

$$\frac{Q, \Gamma \vdash \Delta}{\bar{P}, \Gamma \vdash \Delta} \text{ FC}$$

FORWARD CHAINING

[Simpson, Negri & von Plato, Ciabattoni et al.]

IN GENERAL!

HORN CLAUSES:

$$\forall \bar{z}.(P_1 \wedge P_2 \wedge \dots \wedge P_n \supset Q)$$

IN GENERAL!

HORN CLAUSES:

$$\forall \bar{z}. (P_1 \wedge P_2 \wedge \dots \wedge P_n \supset Q)$$

$$\frac{\Gamma \vdash P_1, \Delta \quad \dots \quad \Gamma \vdash P_n, \Delta}{\Gamma \vdash Q, \Delta}\ BC$$

BACK-CHAINING

[Viganò]

IN GENERAL!

ALSO:

- GEOMETRIC AXIOMS ✓
- CO-GEOMETRIC AXIOMS ✓
- UNIVERSAL AXIOMS ✓
- IMPLEMENTATION ✓
- CUT-ELIMINATION ✓

QUESTION

CAN THIS APPROACH BE USED IN THE SETTING OF

DEFINITE DESCRIPTIONS ?

IS THERE LIFE BEYOND N_2?

$$\forall x. (\neg P \vee \neg\neg P)$$

IS THERE LIFE BEYOND N_2?

$$\forall x.\,(\neg P \vee^{+} \neg\neg P)$$

$$\Gamma \quad \Downarrow \forall x.\,(\neg P \vee^{+} \neg\neg P) \vdash C$$

IS THERE LIFE BEYOND N_2?

$$\forall x.\,(\neg P \vee^{+} \neg\neg P)$$

$$\frac{\Gamma \Downarrow \neg P \vee^{+} \neg\neg P \vdash C}{\Gamma \Downarrow \forall x.\,(\neg P \vee^{+} \neg\neg P) \vdash C}\ \forall_L$$

IS THERE LIFE BEYOND N_2?

$$\forall x.(\neg P \vee^{+} \neg\neg P)$$

$$\dfrac{\dfrac{\Gamma \Uparrow \neg P \vee^{+} \neg\neg P \vdash C}{\Gamma \Downarrow \neg P \vee^{+} \neg\neg P \vdash C}\,\Uparrow}{\Gamma \Downarrow \forall x.(\neg P \vee^{+} \neg\neg P) \vdash C}$$

IS THERE LIFE BEYOND N_2?

$$\forall x.\,(\neg P \vee^{+} \neg\neg P)$$

$$\dfrac{\dfrac{\dfrac{\Gamma \Uparrow \neg P \vdash C \qquad \Gamma \Uparrow \neg\neg P \vdash C}{\Gamma \Uparrow \neg P \vee^{+} \neg\neg P \vdash C}\;\vee^{+}_{L}}{\Gamma \Downarrow \neg P \vee^{+} \neg\neg P \vdash C}}{\Gamma \Downarrow \forall x.\,(\neg P \vee^{+} \neg\neg P) \vdash C}$$

IS THERE LIFE BEYOND N_2?

$$\forall x.\,(\neg P \vee^{+} \neg\neg P)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \neg P \Uparrow \cdot \vdash C}{\Gamma \Uparrow \neg P \vdash C}\,s \qquad \dfrac{\Gamma, \neg\neg P \Uparrow \cdot \vdash C}{\Gamma \Uparrow \neg\neg P \vdash C}\,s}{\Gamma \Uparrow \neg P \vee^{+} \neg\neg P \vdash C}}{\Gamma \Downarrow \neg P \vee^{+} \neg\neg P \vdash C}}{\Gamma \Downarrow \forall x.\,(\neg P \vee^{+} \neg\neg P) \vdash C}$$

IS THERE LIFE BEYOND N_2?

$$\forall x. (\neg P \vee^+ \neg\neg P)$$

NON ATOMIC :(

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \neg P \Uparrow \cdot \vdash C}{\Gamma \Uparrow \neg P \vdash C}\,s \qquad \dfrac{\Gamma, \neg\neg P \Uparrow \cdot \vdash C}{\Gamma \Uparrow \neg\neg P \vdash C}\,s}{\Gamma \Uparrow \neg P \vee^+ \neg\neg P \vdash C}}{\Gamma \Downarrow \neg P \vee^+ \neg\neg P \vdash C}}{\Gamma \Downarrow \forall x. (\neg P \vee^+ \neg\neg P) \vdash C}$$

IS THERE LIFE BEYOND N_2?

$$\forall x.\,(\neg P \vee^+ \neg\neg P)$$

$$\frac{\Gamma, \neg P \Uparrow \cdot \vdash C \qquad \Gamma, \neg\neg P \Uparrow \cdot \vdash C}{\Gamma \Downarrow \forall x.\,(\neg P \vee^+ \neg\neg P) \vdash C}$$

IS THERE LIFE BEYOND N_2?

$$\forall x.\,(\neg P \vee^{+} \neg\neg P)$$

$$\dfrac{\begin{array}{c}\Gamma' \Downarrow \neg P \vdash C' \\ \vdots \\ \Gamma \vdash C\end{array} \qquad \begin{array}{c}\Gamma'' \Downarrow \neg\neg P \vdash C'' \\ \vdots \\ \Gamma \vdash C\end{array}}{\Gamma \vdash C}$$

IS THERE LIFE BEYOND N_2?

$$\forall x.\,(\neg P \vee^{+} \neg\neg P)$$

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{}{\Gamma' \vdash P \Downarrow}}{\Gamma' \Downarrow \neg P \vdash C'} \\ \vdots \\ \Gamma \vdash C\end{array} \qquad \begin{array}{c}\dfrac{\Gamma'' \Uparrow P \vdash \cdot}{\Gamma'' \Downarrow \neg\neg P \vdash C''} \\ \vdots \\ \Gamma \vdash C\end{array}}{\Gamma \vdash C}$$

IS THERE LIFE BEYOND N_2?

$$\forall x. (\neg P \vee^{+} \neg\neg P)$$

$$\dfrac{\begin{array}{c} \overline{P, \Gamma' \vdash \bullet} \\ \vdots \\ \Gamma \vdash C \end{array} \qquad \begin{array}{c} \dfrac{\Gamma'', P \vdash \bullet}{\Gamma'' \vdash C''} \\ \vdots \\ \Gamma \vdash C \end{array}}{\Gamma \vdash C}$$

IS THERE LIFE BEYOND N_2?

$$\forall x. (\neg P \vee^{+} \neg\neg P)$$

SYSTEMS OF RULES [Negri]

$$\dfrac{\begin{array}{c} \dfrac{}{P, \Gamma' \vdash \bullet} \\ \vdots \\ \Gamma \vdash C \end{array} \qquad \qquad \begin{array}{c} \dfrac{\Gamma'', P \vdash \bullet}{\Gamma'' \vdash C''} \\ \vdots \\ \Gamma \vdash C \end{array}}{\Gamma \vdash C}\; r$$

$\mathcal{N}_3$ = HYPERSEQUENTS !

$\alpha = \vee^+ n_i$, $1 \le i \le k$ with $n_i \in \mathcal{N}_2$:

Negri

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\nabla \text{ bipole} \quad \dots \quad \nabla \text{ bipole}}{\{\Downarrow n_i \vdash\}}}{\{n_i \Uparrow \;\vdash\}}\ \text{store}}{\{\Uparrow n_i \vdash\}}\ \vee^+}{\Uparrow \; \vee^+ n_i \;\vdash}\ \text{release}}{\Downarrow \; \vee^+ n_i \;\vdash}$$

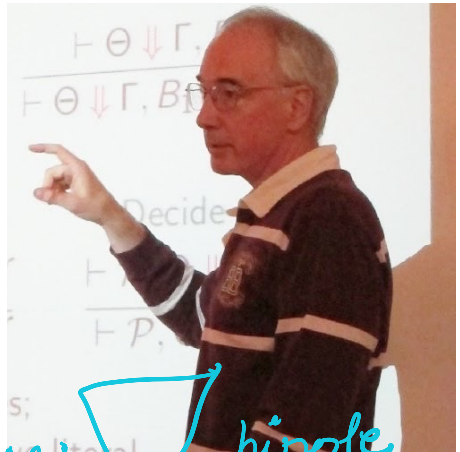
Ciabattoni + Genco

$$\dfrac{\dfrac{\nabla \text{ bipole} \quad \dots \quad \nabla \text{ bipole}}{\{ \,|\, \Rightarrow n_i \}}}{\Rightarrow \vee^+ n_i}$$

N_3 = HYPERSEQUENTS !

DALE

$\alpha = \vee^+ n_i \,, \quad 1 \le i \le K$ with $n_i \in N_2$:



bipole ... bipole

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\Downarrow n_i \vdash\}}{\{n_i \Uparrow \ \vdash\}}\ \text{store}}{\{\Uparrow n_i \vdash\}}\ \vee^+}{\Uparrow \ \vee^+ n_i \ \vdash}\ \text{release}}{\Downarrow \ \vee^+ n_i \vdash}$$

bipole ... bipole

$$\dfrac{\{ \mid \Rightarrow n_i \}}{\Rightarrow \vee^+ n_i}$$

Dale + Elaine

1st order / focusing

ECUMENICAL SYSTEMS

PRAWITZ: GROUND LANGUAGE FOR EMBRACING CONFLICTIVE LOGICS

KUROKAWA: HYPERSEQUENT FOR IPL + CLASSICAL ATOMS

CIABATTONI + GENCO: TRANSFORM Z-SYSTEMS INTO NATURAL DEDUCTION

VON PLATO: NATURAL DEDUCTION IPL + CLASSICAL ATOMS

LUIZ CARLOS + MYSELF: MIX EVERYTHING :)

THE MAIN IDEA

LUIZ CARLOS

$X \vee \neg X$ $\rightsquigarrow$ X A PROPOSITIONAL CLASSICAL VARIABLE

$$\frac{G \mid \Gamma, X \vdash A}{G \mid \Gamma \vdash A \mid X \vdash} \;(1)$$

[Kurokawa]

THE MAIN IDEA

$$
\cfrac{\cfrac{\cfrac{\Gamma \Uparrow P \vdash C \qquad \cfrac{\cfrac{\overline{\Gamma' \vdash P \Downarrow}}{\vdots} \\ \Gamma, \neg P \Uparrow \vdash C}{\Gamma \Uparrow \neg P \vdash C}}{\Gamma \Uparrow P \vee^{+} \neg P \vdash C}}{\Gamma \Downarrow P \vee^{+} \neg P \vdash C}}{\Gamma \Downarrow \forall x.\, P \vee^{+} \neg P \vdash C}
$$

THE MAIN IDEA

$$\dfrac{\Gamma, P \vdash C \qquad \begin{array}{c} \overline{\quad \Gamma', P \vdash C' \quad} \\ \vdots \\ \Gamma \vdash C \end{array}}{\Gamma \vdash C}$$

THE MAIN IDEA

$$\dfrac{\Gamma, P \vdash C \qquad \begin{array}{c}\dfrac{}{\Gamma', P \vdash C'}\\ \vdots\\ \Gamma \vdash C\end{array}}{\Gamma \vdash C}$$

$\Downarrow$ [Ciabattoni + Genco]

$$\dfrac{G \mid \Gamma, P \vdash A}{G \mid \Gamma \vdash A \mid \Gamma', P \vdash B}$$

THE MAIN IDEA

$$\dfrac{\Gamma, p \vdash C \qquad \begin{array}{c} \dfrac{}{\Gamma', p \vdash C'} \\ \vdots \\ \Gamma \vdash C \end{array}}{\Gamma \vdash C}$$

$\Downarrow$ [Ciabattoni + Genco]

$$\dfrac{G \mid \Gamma, p \vdash A}{G \mid \Gamma \vdash A \mid \Gamma', p \vdash B}\,(1_h) \qquad \dfrac{G \mid \Gamma, x \vdash A}{G \mid \Gamma \vdash A \mid x \vdash}\,(1)$$

[Kurokawa]

THE MAIN IDEA

$$\frac{G \mid \Gamma, p \vdash A}{G \mid \Gamma \vdash A \mid \Gamma', p \vdash B}\ (1_h)$$

$\Downarrow$ [Ciabattoni + Genco]

$$\frac{\begin{array}{c}[p] \\ \vdots \\ C\end{array} \qquad \begin{array}{c}[\neg p] \\ \vdots \\ C\end{array}}{C}\ \mathrm{Em}$$

[von Plato]

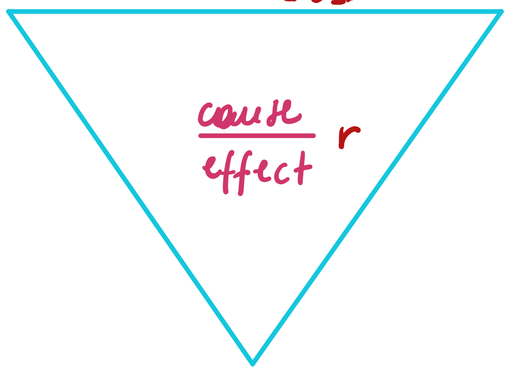
MATHEMATICAL EXPLANATIONS

FRANCESCA

POGGIOLESI:

PROOFS = TREES

$U_1 \dots U_n \quad [D_1] \dots [D_m]$



THEOREM

U_i = GROUNDS $\sim$ REASONS WHY THE THEOREM IS TRUE
(undischarged assump.)

r = GENERALIZATION (given by axioms)

MATHEMATICAL EXPLANATIONS

Theorem. The sum of the angles of any quadrangle is 360°

MATHEMATICAL EXPLANATIONS

Theorem. The sum of the angles of any quadrangle is 360°

$$\frac{\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \rightarrow Sq360(xyzw)}{\forall xyzw.\ Qu(xyzw) \rightarrow Sq360(xyzw)}$$

MATHEMATICAL EXPLANATIONS

<u>Theorem</u>. The sum of the angles of any quadrangle is 360°

$$
\begin{array}{c}
\vdots \\
\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \to St180(xyz) \wedge St180(yzw) \\
\hline
\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \to Sq360(xyzw) \\
\hline
\forall xyzw.\ Qu(xyzw) \to Sq360(xyzw)
\end{array}
$$

MATHEMATICAL EXPLANATIONS

Theorem. The sum of the angles of any quadrangle is 360°

$$\vdots$$

$$\frac{\dfrac{\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \to St180(xyz) \wedge St180(yzw)}{\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \to Sq360(xyzw)}}{\forall xyzw.\ Qu(xyzw) \to Sq360(xyzw)}$$

WHOLE NEW APPROACH!

MATHEMATICAL EXPLANATIONS

CONDITIONAL AXIOMS:

$$\forall \bar{x}.\ H(\bar{x}) \to (F(\bar{x}) \leftrightarrow G(\bar{x}))$$

MATHEMATICAL EXPLANATIONS

CONDITIONAL AXIOMS:

$$\forall \bar{x}.\ H(\bar{x}) \to (F(\bar{x}) \leftrightarrow G(\bar{x}))$$

condition

body

defined axiom

MATHEMATICAL EXPLANATIONS

$$\frac{\Gamma, H(t), F(t) \Rightarrow \Delta}{\Gamma, H(t), G(t) \Rightarrow \Delta}$$

$$\frac{\Gamma, H(t) \Rightarrow \Delta, F(t)}{\Gamma, H(t) \Rightarrow \Delta, G(t)}$$

CONDITIONAL AXIOMS:

$$\forall \bar{x}.\ H(\bar{x}) \rightarrow (F(\bar{x}) \leftrightarrow G(\bar{x}))$$

MATHEMATICAL EXPLANATIONS

CONDITIONAL AXIOMS:

$$\forall \bar{x}.\ H(\bar{x}) \rightarrow (F(\bar{x}) \leftrightarrow G(\bar{x}))$$

$$\frac{\Gamma, H(t), F(t) \Rightarrow \Delta}{\Gamma, H(t), G(t) \Rightarrow \Delta}$$

$$\frac{\Gamma, H(t) \Rightarrow \Delta, F(t)}{\Gamma, H(t) \Rightarrow \Delta, G(t)}$$

SPECIAL CONDITIONS FOR:

* INVERTIBILITY
* CUT ELIMINATION

MATHEMATICAL EXPLANATIONS

CONDITIONAL AXIOMS:

$$\forall \bar{x}.\ H(\bar{x}) \to (F(\bar{x}) \leftrightarrow G(\bar{x}))$$

IMPLEMENTED IN THE L-FRAMEWORK

SPECIAL CONDITIONS FOR:

* INVERTIBILITY
* CUT ELIMINATION

MATHEMATICAL EXPLANATIONS

Theorem. The sum of the angles of any quadrangle is $360°$

(i) $\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \rightarrow$

$St180(xyz) \wedge St180(yzw) \longleftrightarrow Sq360(xyzw)$

(ii) $\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \leftrightarrow Qu(xyzw)$

MATHEMATICAL EXPLANATIONS

<u>Theorem</u>. The sum of the angles of any quadrangle is 360°

(i) $\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \rightarrow$

$St180(xyz) \wedge St180(yzw) \longleftrightarrow Sq360(xyzw)$

(ii) $\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \leftrightarrow Qu(xyzw)$

$$\frac{Tr(xyz) \wedge Tr(yzw) \rightarrow Sq360(xyzw)}{Qu(xyzw) \rightarrow Sq360(xyzw)}$$

MATHEMATICAL EXPLANATIONS

<u>Theorem</u>. The sum of the angles of any quadrangle is 360°

(i) $\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \rightarrow St180(xyz) \wedge St180(yzw) \leftrightarrow Sq360(xyzw)$

(ii) $\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \leftrightarrow Qu(xyzw)$

DEEP INFERENCE

$$\frac{\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \rightarrow Sq360(xyzw)}{\forall xyzw.\ Qu(xyzw) \rightarrow Sq360(xyzw)}$$

MATHEMATICAL EXPLANATIONS

Theorem. The sum of the angles of any quadrangle is 360°

$$\frac{\forall xyzw.\ Tr(xyz) \wedge Tr(yzw) \to Sq360(xyzw)}{\forall xyzw.\ Qu(xyzw) \to Sq360(xyzw)}$$

PROFOUND - INTUITIONISTIC TOOL (KAUSTUV CHAUDHURI)

```
Typing:
i : \type.
tr : i -> i -> i -> \o.
qu : i -> i -> i -> i -> \o.
sq360 : i -> i -> i -> i -> \o.

Program:
(\A [x, y, z, w] qu x y z w \to (tr x y z \and tr y z w)) \to
(\A [x, y, z, w] (tr x y z \and tr y z w \to sq360 x y z w) \to
qu x y z w \to sq360 x y z w)
```

CONCLUDING ...

- DYCKHOFF + NEGRI: EVERY CLASSICAL THEORY HAS A GEOMETRIC CONSERVATIVE EXTENSION
- LITERATURE: AD HOC, PROPOSITIONAL, LABEL, STRUCTURAL ...
- MODAL SYSTEMS

CONCLUDING ...

- OUR APPROACH : SYSTEMATIC & MODULAR
- RULES FOR AXIOMS
- BIPOLAR = BIPOLE
- CLASSICAL + INTUITIONISTIC
- CUT ELIMINATION
- META - LEVEL REASONING

¡GRACIAS!

THANK YOU!

OBRIGADA!