From Second-order Quantification to Second-order Definite Descriptions

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The main idea of the talk

- Definite descriptions are typically first-order expressions of the structure $\iota x \varphi$, constructed using a ι-term forming operator, an individual variable x, and a formula φ.
- The intuitive interpretation of definite descriptions is as follows: the unique object x for which the formula φ is true.
- This presentation aims to introduce a second-order generalization of definite descriptions: $\iota X \varphi$, where X denotes a relational variable.
- Expressions of this form indicate the unique *relation* such that the formula φ is true.
- We intend to mainly discuss Russell-style approach to first-order definite descriptions and introduce its second-order version. However, we plan to talk about Frege-style approach as well.
- While full second-order logic is incomplete, its fragment defined by Henkin's general models admits completeness. We develop our theory within this fragment and formalize it using a cut-free sequent calculus.

- First-order quantification is defined over objects, while second-order one over their properties and relations between them.
- We utilize this relationship between first-order and second-order quantifiers when introducing second-order definite descriptions.
- It is noteworthy that prior efforts have been made to examine definite descriptions in the realm of second- or higher-order logics, as conducted by Makarenko and Benzmüller (2020). Nonetheless, in their analysis, definite descriptions continue to function as first-order expressions pertaining to objects.

Makarenko, I., Benzmüller, C.: Positive Free Higher-Order Logic and Its Automation via a Semantical Embedding. In: Schmid, U., Klügl, F., Wolter, D. (eds.) KI 2020: Advances in Artificial Intelligence. KI 2020, LNCS, vol. 12325, pp. 116–131. Springer, Cham (2020)

Let us present several examples of the second-order definite descriptions

- The term $\iota R \forall x \forall y \, (R(x, y) \leftrightarrow x < y)$ denotes the unique binary relation that captures the "less than" ordering on a domain.
- Similarly, $\iota P \forall x \, (P(x) \leftrightarrow \mathrm{Prime}(x))$ denotes the unique property shared by all prime numbers.
- The relation between objects $a, b, c_1, \ldots, c_n$ such that a is closer to b than $c_1, \ldots, c_n$.
- The property of being the square root of 16.
- The property of being the last dinosaur alive today.

- Russellian theory of definite descriptions (Russell, 1905; Whitehead, Russell, 1910) is arguably one of the most recognized and frequently accepted, notwithstanding the criticism it has received. It possesses several disadvantages; nevertheless, there are methods to mitigate at least some of them.
- We follow the presentation of Russellian theory as articulated by Indrzejczak and Zawidzki (2023) and Indrzejczak and Kürbis (2023).

- Russell, B.: On denoting. Mind **14**, 479–493 (1905)
- Whitehead, A.N., Russell, B.: Principia Mathematica, vol. I. Cambridge University Press, Cambridge (1910)
- Indrzejczak, A., Zawidzki, M.: When Iota meets Lambda. Synthese **201**(2), 1–33 (2023)
- Indrzejczak, A., Kürbis, N.: A Cut-Free, Sound and Complete Russellian Theory of Definite Descriptions. In: Ramanayake, R., Urban, J. (eds.) Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2023, LNCS, vol. 14278, pp. 112–130. Springer, Cham (2023)

In Russellian theory, definite descriptions are characterized by the following formula, where ψ must be limited to atomic formulas, unless additional mechanisms are introduced to mark scope distinctions:

$$\psi(\iota y\varphi) \leftrightarrow \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi)$$

Indrzejczak, Zawidzki, and Kürbis' approach characterize them with the help of λ-operator as follows:

$$(\lambda x\psi)\iota y\varphi \leftrightarrow \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi)$$

- This approach allows both complex and primitive predicates to be applied to definite descriptions while avoiding scope-related issues.
- In particular, it helps us answer the question: in the negated expression $\neg\psi(\iota y\varphi)$, does the negation apply to the entire expression, or solely to ψ?
- Whitehead and Russell, seeing the issue, proposed the method of scope distinctions. Nonetheless, it is recognized for its clumsiness.
- One may attempt to circumvent the issue by limiting the formulas to atomic ones; however, this considerably diminishes the expressive capacity of Russellian theory.
- Another concept has been proposed by Kürbis. He implements a binary quantifier represented as $Ix[\varphi, \psi]$. This resolves the issues; but, if one wants to consider definite descriptions as terms, an alternative approach is required.
- This inspired Indrzejczak, Zawidzki, and Kürbis to apply predicate abstracts of the form $\lambda x\varphi$ ('the property of being φ') to terms, including definite descriptions, to obtain formulas called lambda atoms.

- Another source for motivation for the introduction of λ is that if ψ is complex in the Russelian formula characterizing definite descriptions, one may readily encounter a contradiction.
- The application of λ addresses this problem, while a similar outcome could be attained through the adoption of free logic (rendering the entire theory deductively weaker and incapable of inferring contradictions; see, e.g. (Indrzejczak, 2020)) or the utilization of paraconsistent logic (Petrukhin, 2024).

Indrzejczak, A.: Free Definite Description Theory – Sequent Calculi and Cut Elimination. Logic and Logical Philosophy **29**(4), 505–539 (2020)

Petrukhin, Y.: A binary quantifier for definite descriptions in Nelsonian free logic. In: Indrzejczak, A., Zawidzki, M. (eds.) Proceedings Eleventh International Conference on Non-Classical Logics. Theory and Applications, EPTCS, vol. 415, pp. 5–15 (2024)

- This study adheres to the framework established by Indrzejczak and Kürbis (2023) in presenting the semantics and employs their cut-free sequent calculus, which is adequate with respect to this semantics.
- We extend their approach by incorporating second-order quantifiers, a second-order variant of identity, and ultimately a second-order interpretation of lambda terms and definite descriptions.
- Unlike them, we do not provide a constructive proof of the cut admissibility theorem, as we believe this subject, due to its complexity, requires a separate paper.
- The cut admissibility theorem for second- and higher-order logics has remained an unresolved issue in proof theory for an extended duration, referred to as Takeuti's conjecture (Takeuti, 1953). Various scholars, utilizing distinct methodologies, reached a positive resolution: Tait (1966), Prawitz (1968), Takahashi (1967), Girard (1971). Developing a syntactic constructive proof remains an open problem.

- However, we provide a semantic proof of this statement obtained as a consequence of a Hintikka-style completeness proof in the spirit of [1, 2].
- Let us start with a description of the logic **RL** from (Indrzejczak, Zawidzki, 2023; Indrzejczak, Kürbis, 2023), which corresponds to the formulation of the Russellian theory involving λ discussed above.

Avron, A., Lahav, O.: A simple cut-free system for a paraconsistent logic equivalent to S5. In: Advances in Modal Logic, vol. 12, pp. 29–42. College Publications (2018)

Lahav, O., Avron, A.: A semantic proof of strong cut-admissibility for first-order Gödel logic. Journal of Logic and Computation **23**(1), 59–86 (2013)

Languages and Semantics

- *The language* $\mathscr{L}^1$ *of* **RL** is a standard first-order language with identity and without function symbols, but supplied with λ and ι. It does not have constant symbols, but such constants are introduced in the completeness proof.
- The language is built from two disjoint sets of symbols: VAR, representing variables, and PAR, representing parameters.
- In the proof-theoretic framework of **RL**, elements of VAR are used exclusively as bound variables, while PAR provides the symbols for free variables.
- In contrast, the semantic framework does not enforce this distinction.
- The basic terms of the language consist of variables and parameters.
- Additionally, we allow expressions formed using the definite description operator ι applied to predicate abstracts. These are referred to as quasi-terms.

"We mention only the following formation rules for the more general notion of a formula used in the semantics:

- If P^n is a predicate symbol (including $=$) and $t_1, \ldots, t_n \in VAR \cup PAR$, then $P^n(t_1, \ldots, t_n)$ is a formula (atomic formula).
- If φ is a formula, then $(\lambda x \varphi)$ is a predicate abstract.
- If φ is a formula, then $\iota x \varphi$ is a quasi-term.
- If φ is a predicate abstract and t a term or quasi-term, then φt is a formula (lambda atom)." (Indrzejczak, Kürbis, 2023, p. 115)

Indrzejczak, A., Kürbis, N.: A Cut-Free, Sound and Complete Russellian Theory of Definite Descriptions. In: Ramanayake, R., Urban, J. (eds.) Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2023, LNCS, vol. 14278, pp. 112–130. Springer, Cham (2023)

- We write $\mathscr{F}^1$ for the set of all formulas of $\mathscr{L}^1$,
- $x, y, z, x_1, \ldots$ for the members of VAR,
- $a, b, c, a_1, \ldots$ for the elements of PAR,
- φ^x_t for the result of replacing x by t in φ, and, similarly, $\varphi^{x_1,\ldots,x_n}_{t_1,\ldots,t_n}$ for the result of a simultaneous replacing $x_1, \ldots, x_n$ by $t_1, \ldots, t_n$ in φ.
- When t is a variable y, we assume that y is free for x in φ, meaning that the substitution does not cause any formerly free occurrence of y to become bound within φ.
- We presume a similar condition for $\varphi^{x_1,\ldots,x_n}_{t_1,\ldots,t_n}$.

The semantics of **RL**. (Indrzejczak, Kürbis, 2023, p. 115, 116)

"A *model* is a structure $M = \langle D, I \rangle$, where for each n-argument predicate P^n, $I(P^n) \subseteq D^n$. An *assignment* v is a function $v : VAR \cup PAR \to D$. An *x-variant* v' of v agrees with v on all arguments, save possibly x. We write v^x_o to denote the x-variant of v with $v^x_o(x) = o$. The notion of *satisfaction* of a formula φ *with* v, in symbols $M, v \models \varphi$, is defined as follows, where $t \in VAR \cup PAR$:

$$M, v \models P^n(t_1, \ldots, t_n) \text{ iff } \langle v(t_1), \ldots, v(t_n) \rangle \in I(P^n)$$

$$M, v \models t_1 = t_2 \text{ iff } v(t_1) = v(t_2)$$

$$M, v \models (\lambda x \psi)t \text{ iff } M, v^x_o \models \psi, \text{ where } o = v(t)$$

$$M, v \models (\lambda x \psi)\iota y \varphi \text{ iff there is an } o \in D \text{ such that } M, v^x_o \models \psi,$$
$$M, v^x_o \models \varphi^y_x, \text{ and for any } y\text{-variant } v' \text{ of } v^x_o,$$
$$\text{if } M, v' \models \varphi, \text{ then } v'(y) = o$$

$$M, v \models \neg\varphi \text{ iff } M, v \not\models \varphi$$

$$M, v \models \varphi \wedge \psi \text{ iff } M, v \models \varphi \text{ and } M, v \models \psi$$

$$M, v \models \forall x \varphi \text{ iff } M, v^x_o \models \varphi, \text{ for all } o \in D."$$

- The truth conditions for $\vee$, $\to$, $\leftrightarrow$, and $\exists$ are standard.
- If one adds constants to $\mathscr{L}^1$, then we postulate that $I(k) \in D$, for each constant k.
- The notions of satisfiable and valid formulas are defined in a standard way.
- The consequence relation is understood as follows, for all sets of $\mathscr{L}^1$-formulas Γ and $\mathscr{L}^1$-formula A:
  $\Gamma \models_{\mathbf{RL}} \varphi$ iff in every model M and every assignment v, if $M, v \models \psi$, for all $\psi \in \Gamma$, then $M, v \models \varphi$.

Let us now describe the logic $\mathbf{RL^2}$, a second-order generalization of **RL**.

- *The language* $\mathscr{L}^2$ *of* $\mathbf{RL^2}$ is a second-order extension of $\mathscr{L}^1$.
- In addition to individual variables and parameters, we have the sets
- $VAR^2 = \{X, Y, Z, X_1, \ldots\}$ and
- $PAR^2 = \{A, B, C, A_1, \ldots\}$ of n-ary relational variables and parameters, respectively (unary ones might be called property variables and parameters).
- As in the first-order case, this distinction is important for proof theory, but might be relaxed in the case of semantics.
- The terms are constants (if added) and individual variables/parameters.
- Notice that relational variables/parameter are not terms.
- Atomic formulas are as follows: $t_1 = t_2$, $P(t_1, \ldots, t_n)$, $X = Y$, and $X(t_1, \ldots, t_n)$, if $t_1, \ldots, t_n$ are terms, P is an n-ary relation symbol, and X and Y are n-ary relational variables/parameters.

- The formula $X = Y$ is understood as $\forall x_1 \ldots \forall x_n \big(X(x_1, \ldots, x_n) \leftrightarrow Y(x_1, \ldots, x_n)\big)$. The formula $t_1 = t_2$ might be defined as $\forall X(X(t_1) \leftrightarrow X(t_2))$.
- In addition to the above described atomic and first-order formulas, we define the following ones:
  - If φ is a formula and $X \in VAR^2$, then $\forall X\varphi$ and $\exists X\varphi$ are formulas.
  - If φ is a formula, then $(\lambda X\varphi)$ is a *relational* abstract.
  - If φ is a formula, then $\iota X\varphi$ is a *pseudo*-term.
  - If φ is a relational abstract and t is a pseudo-term, then φt is a formula.
- We write $\mathscr{F}^2$ for the set of all formulas of $\mathscr{L}^2$, φ^X_P for the result of replacing X by a predicate symbol P in φ.

The first version of the semantics (without a complete calculus).

- In a model $M = \langle D, I \rangle$, an assignment v should be redefined as follows: $v(x) \in D$, for $x \in VAR \cup PAR$, and $v(X) \subseteq D^n$, for $X \in VAR^2 \cup PAR^2$,
- An *X-variant* v' of v agrees with v on all arguments, save possibly X. We write v^X_O to denote the X-variant of v with $v^X_O(X) = O$, where $O \subseteq D^n$.
- The definition of the notion of satisfaction of a formula φ with v is extended by the following cases, where $t \in VAR \cup PAR$:

$$M, v \models X(t_1, \ldots, t_n) \text{ iff } \langle v(t_1), \ldots, v(t_n) \rangle \in v(X), \text{ if } X \text{ is } n\text{-ary},$$

$$M, v \models X = Y \text{ iff } v(X) = v(Y),$$

$$M, v \models (\lambda X \psi)\iota Y \varphi \text{ iff there is an } O \subseteq D^n \text{ such that } M, v^X_O \models \psi,$$
$$M, v^X_O \models \varphi^Y_X, \text{ and for any } Y\text{-variant } v' \text{ of } v^X_O,$$
$$\text{if } M, v' \models \varphi, \text{ then } v'(Y) = O$$

$$M, v \models \forall X \varphi \text{ iff } M, v^X_O \models \varphi, \text{ for all } O \subseteq D^n,$$

$$M, v \models \exists X \varphi \text{ iff } M, v^X_O \models \varphi, \text{ for some } O \subseteq D^n.$$

- This semantics lacks the completeness theorem.
- In order to obtain this theorem, we need to deal with a fragment of the second-order logic: we should restrict the interpretations of the relational variables/parameters.
- Henkin's concept of a general model will help us with this issue.

Henkin, L.: Completeness in the Theory of Types. J. Symbolic Logic **15**(2), 81–91 (1950)

The second version of the semantics (with a complete calculus). The semantics of $\mathbf{RL^2}$.

- Although second-order logic is known to be incomplete, there exists a fragment that is complete with respect to general models.
- A *general model* is a pair $\mathfrak{M} = \langle M, G \rangle$, where $M = \langle D, I \rangle$ is a model and G is a set of subsets, relations (of any arity) on D.
- Notice that $v(X) \in G \subseteq \mathcal{P}(D^n)$.
- We define the notion of satisfaction of a formula φ with v in a general model, symbolically $\mathfrak{M}, v \models \varphi$, for second-order formulas as follows:

$$\mathfrak{M}, v \models X(t_1, \ldots, t_n) \text{ iff } \langle v(t_1), \ldots, v(t_n) \rangle \in v(X), \quad \text{if } X \text{ is } n\text{-ary},$$

$$\mathfrak{M}, v \models X = Y \text{ iff } v(X) = v(Y),$$

$$\mathfrak{M}, v \models (\lambda X \psi)\iota Y \varphi \text{ iff there is an } O \in G \text{ such that } \mathfrak{M}, v^X_O \models \psi, \; \mathfrak{M}, v^X_O \models \varphi^Y_X, \text{ and for any } Y\text{-variant } v' \text{ of } v^X_O, \text{ if } \mathfrak{M}, v' \models \varphi, \text{ then } v'(Y) = O$$

$$\mathfrak{M}, v \models \forall X \varphi \text{ iff } \mathfrak{M}, v^X_O \models \varphi, \text{ for all } O \in G,$$

$$\mathfrak{M}, v \models \exists X \varphi \text{ iff } \mathfrak{M}, v^X_O \models \varphi, \text{ for some } O \in G.$$

- The notions of a satisfiable formula and a valid formula are defined in a standard way.
- The consequence relation is defined as follows, for all $\Gamma \subseteq \mathscr{F}^2$ and $A \in \mathscr{F}^2$:
  $\Gamma \models_{\mathbf{RL^2}} \varphi$ iff in every general model $\mathfrak{M}$ and every assignment v, if $\mathfrak{M}, v \models \psi$, for all $\psi \in \Gamma$, then $\mathfrak{M}, v \models \varphi$.
- In the completeness proof, we extend the language by individual constants $k, k_1, k_2, \ldots$, and by *relational constants* $K, K_1, K_2, \ldots$.
- These relational constants are introduced to represent fixed n-ary relations over individual constants.
- Formally, each relational constant K^n corresponds to a set of n-tuples of individual constants, and is interpreted as an element of the general model domain $G \subseteq \mathcal{P}(D^n)$.
- By this concept, we understand the elements of G to be syntactically named via relational constants, allowing us to treat them as concrete surrogates for second-order values during the construction of canonical models.

Sequent calculi. Kürbis and Indrzejczak's calculus for **RL**

$$\text{(Cut)}\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad \text{(AX)}\ \varphi \Rightarrow \varphi$$

$$\text{(W}\Rightarrow\text{)}\ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\text{W})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \qquad \text{(C}\Rightarrow\text{)}\ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\text{C})\ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \qquad (\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\wedge \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$$

$$(\leftrightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi \quad \varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \leftrightarrow \psi, \Gamma \Rightarrow \Delta} \qquad (\forall \Rightarrow)\ \frac{\varphi^x_b, \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\leftrightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \leftrightarrow \psi} \qquad (\Rightarrow \forall)\ \frac{\Gamma \Rightarrow \Delta, \varphi^x_a}{\Gamma \Rightarrow \Delta, \forall x\varphi}$$

$$(\exists \Rightarrow)\ \frac{\varphi^x_a, \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \exists)\ \frac{\Gamma \Rightarrow \Delta, \varphi^x_b}{\Gamma \Rightarrow \Delta, \exists x\varphi} \qquad (=+)\ \frac{b = b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(=-)\ \frac{\mathscr{A}^x_c, \Gamma \Rightarrow \Delta}{b = c, \mathscr{A}^x_b, \Gamma \Rightarrow \Delta} \qquad (\lambda \Rightarrow)\ \frac{\psi^x_b, \Gamma \Rightarrow \Delta}{(\lambda x\psi)b, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \lambda)\ \frac{\Gamma \Rightarrow \Delta, \psi^x_b}{\Gamma \Rightarrow \Delta, (\lambda x\psi)b}$$

where a is a fresh parameter (*Eigenvariable*), not present in Γ, Δ and φ, whereas b, c are arbitrary parameters. $\mathscr{A}$ in $(=-)$ is an atomic formula.

$$(\iota_1 \Rightarrow)\ \frac{\varphi^y_a, \psi^x_a, \Gamma \Rightarrow \Delta}{(\lambda x \psi)\iota y \varphi, \Gamma \Rightarrow \Delta} \quad (\iota_2 \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi^y_b \quad \Gamma \Rightarrow \Delta, \varphi^y_c \quad b = c, \Gamma \Rightarrow \Delta}{(\lambda x \psi)\iota y \varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \iota)\ \frac{\Gamma \Rightarrow \Delta, \varphi^y_b \quad \Gamma \Rightarrow \Delta, \psi^x_b \quad \varphi^y_a, \Gamma \Rightarrow \Delta, a = b}{\Gamma \Rightarrow \Delta, (\lambda x \psi)\iota y \varphi}$$

where a is a fresh parameter (*Eigenvariable*), not present in Γ, Δ and φ, whereas b, c are arbitrary parameters.

Theorem.

Sequent calculus for **RL** is sound, complete, and cut-free.

Indrzejczak, A., Kürbis, N.: A Cut-Free, Sound and Complete Russellian Theory of Definite Descriptions. In: Ramanayake, R., Urban, J. (eds.) Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2023, LNCS, vol. 14278, pp. 112–130. Springer, Cham (2023)

Sequent calculus for $\mathbf{RL^2}$

$$(=^2\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, X^{x_1,\dots,x_n}_{b_1,\dots,b_n}, Y^{x_1,\dots,x_n}_{b_1,\dots,b_n} \qquad X^{x_1,\dots,x_n}_{b_1,\dots,b_n}, Y^{x_1,\dots,x_n}_{b_1,\dots,b_n}, \Gamma \Rightarrow \Delta}{X = Y, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow=^2)\ \frac{X^{x_1,\dots,x_n}_{a_1,\dots,a_n}, \Gamma \Rightarrow \Delta, Y^{x_1,\dots,x_n}_{a_1,\dots,a_n} \qquad Y^{x_1,\dots,x_n}_{a_1,\dots,a_n}, \Gamma \Rightarrow \Delta, X^{x_1,\dots,x_n}_{a_1,\dots,a_n}}{\Gamma \Rightarrow \Delta, X = Y}$$

$$(\exists^2\Rightarrow)\ \frac{\varphi^X_A, \Gamma \Rightarrow \Delta}{\exists X\varphi, \Gamma \Rightarrow \Delta} \quad (\Rightarrow\exists^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^X_B}{\Gamma \Rightarrow \Delta, \exists X\varphi} \quad (\iota^2_1\Rightarrow)\ \frac{\varphi^Y_A, \psi^X_A, \Gamma \Rightarrow \Delta}{(\lambda X\psi)\iota Y\varphi, \Gamma \Rightarrow \Delta}$$

$$(\forall^2\Rightarrow)\ \frac{\varphi^X_B, \Gamma \Rightarrow \Delta}{\forall X\varphi, \Gamma \Rightarrow \Delta} \quad (\iota^2_2\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi^Y_B \quad \Gamma \Rightarrow \Delta, \varphi^Y_C \quad B = C, \Gamma \Rightarrow \Delta}{(\lambda X\psi)\iota Y\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\forall^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^X_A}{\Gamma \Rightarrow \Delta, \forall X\varphi} \quad (\Rightarrow\iota^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^Y_B \quad \Gamma \Rightarrow \Delta, \psi^X_B \quad \varphi^Y_A, \Gamma \Rightarrow \Delta, A = B}{\Gamma \Rightarrow \Delta, (\lambda X\psi)\iota Y\varphi}$$

where $a_1, \dots, a_n$ are fresh individual parameters, not present in Γ, Δ; $b_1, \dots, b_n$ are arbitrary ind. parameters; A is a fresh relational parameter, not present in Γ, Δ, φ; B and C are arb. rel. parameters.

Theorem

Sequent calculus for $\mathbf{RL^2}$ is sound, complete, and cut-free.

Fregean approach to definite descriptions

- Fregean approach might be called the theory of a chosen object, since it assumes an existence of some fixed object which is the denotation of improper definite descriptions.
- This approach may be troublesome from a philosophical point of view (one may ask about the ontological status of this chosen object which is), but it seems to be the simplest one and practically useful.
- As we will see later, the complication with a chosen object finds a natural solution in a second-order case.
- So, the definition of Fregean definite description is as follows:
- If there is a unique $o \in M$ such that $M, v^x_o \models \varphi$, then $I(\iota x \varphi) = o$; otherwise $I(\iota x \varphi) = \mathfrak{i}$, where $\mathfrak{i}$ is a fixed object in M.

This condition can be generalised to the second-order case as follows (recall that $M = \langle D, I \rangle$):

- If there is a unique $O \subseteq D^n$ such that $M, v_O^X \models \varphi$, then $I(\iota X \varphi) = O$; otherwise $I(\iota X \varphi) = \mathfrak{I}$, where $\mathfrak{I}$ is a fixed subset of M.

In a general model (recall that $\mathfrak{M} = \langle M, G \rangle$), it might be formulated as follows:

- If there is a unique $O \in G$ such that $\mathfrak{M}, v_O^X \models \varphi$, then $I(\iota X \varphi) = O$; otherwise $I(\iota X \varphi) = \mathfrak{I}$, where $\mathfrak{I}$ is a fixed element of G.

We may offer a natural specification of the nature of $\mathfrak{J}$. If a second-order definite description is improper because there are no such set O, then we may identify $\mathfrak{J}$ with $\varnothing$. Thus, we can propose the following condition:

- If there is a unique $O \subseteq D^n$ such that $M, v_O^X \models \varphi$, then $I(\iota X \varphi) = O$;
- if there are no such O, then $I(\iota X \varphi) = \varnothing$;
- if there are several sets $O_1, \ldots, O_n$ such that $M, v_{O_i}^X \models \varphi$, then $I(\iota X \varphi) = \mathfrak{J}$.

It might be straightforwardly adapted for general models.

Sequent calculus (sound, complete, and cut-free)

$$(\Rightarrow \iota) \quad \frac{\varphi^x_a, \Gamma \Rightarrow \Delta, t = a \quad t = a, \Pi \Rightarrow \Sigma, \varphi^x_a}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, t = \iota x \varphi}$$

$$(\Rightarrow \mathfrak{i}1) \quad \frac{\varphi^x_a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathfrak{i} = \iota x \varphi}$$

$$(\Rightarrow \mathfrak{i}2) \quad \frac{\Gamma \Rightarrow \Delta, \varphi^x_{t_1} \quad \Pi \Rightarrow \Sigma, \varphi^x_{t_2} \quad t_1 = t_2, \Lambda \Rightarrow \Theta}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \mathfrak{i} = \iota x \varphi}$$

where a is not in $\Gamma, \Delta, \Pi, \Sigma, \varphi$; and $t \neq \mathfrak{i}$.

Indrzejczak, A. Fregean Description Theory in Proof-Theoretical Setting. Logic and Logical Philosophy, 28(1) (2019), 137–155.

Sequent calculus (a preliminary version)

$$(\Rightarrow \iota^2) \quad \frac{\varphi^X_A, \Gamma \Rightarrow \Delta, t = A \quad t = A, \Pi \Rightarrow \Sigma, \varphi^X_A}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, t = \iota X \varphi}$$

$$(\Rightarrow \varnothing) \quad \frac{\varphi^X_A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varnothing = \iota X \varphi}$$

$$(\Rightarrow \mathfrak{J}) \quad \frac{\Gamma \Rightarrow \Delta, \varphi^X_{t_1} \quad \Pi \Rightarrow \Sigma, \varphi^X_{t_2} \quad t_1 = t_2, \Lambda \Rightarrow \Theta}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \mathfrak{J} = \iota X \varphi}$$

where A is not in $\Gamma, \Delta, \Pi, \Sigma, \varphi$; and $t \neq \mathfrak{J}$, $t \neq \varnothing$.

Free theories of definite descriptions. Lambert's approach

The theories of DD in positive or negative free logics are usually based on Lambert axiom (L):

$$\forall y(y = \imath x\varphi \leftrightarrow \forall x(\varphi \leftrightarrow y = x)) \qquad (L)$$

Types of Freedom:

Two criteria:

I Evaluation of atomic formulae with nondenoting terms:

1. positive logics – they may be true;
2. negative logics – all such formulae are evaluated as false (or, to the same effect, all primitive predicates and functions are strict, i.e. interpreted only on denoting terms);
3. neutral free logics (strictly Fregean logic) – no truth value (or the third true value, a gap).

II Domains:

1. inclusive logics – admitting empty domains (universally free logics);
2. noninclusive – no empty domains.

- The language is extended by an existence predicate, $\mathcal{E}$, which is interpreted as follows: $I(\mathcal{E}) \subseteq D$.
- In negative and neutral free logic, the interpretation of $=$ and other predicates is restricted to the elements of $I(\mathcal{E})$.
- The interpretation of quantifiers is restricted to the elements of $I(\mathcal{E})$.
- Lambert-style DD are defined as follow:

$$I(\imath x \varphi) = \begin{cases} a \text{ iff } M, v^x_a \models \varphi \text{ and } M, v^x_b \not\models \varphi \text{ for every } a \neq b \in \mathcal{I}(\mathcal{E}); \\ \text{otherwise it is undefined.} \end{cases}$$

The second-order version might look as follows, $\mathcal{I}(\mathcal{E}) \in G$:

$$I(\imath X \varphi) = \begin{cases} A \text{ iff } M, v_A^X \models \varphi \text{ and } M, v_B^X \not\models \varphi \text{ for every } A \neq B \subseteq \mathcal{I}(\mathcal{E}); \\ \text{otherwise it is undefined.} \end{cases}$$

Conclusion. Subjects for future research

- The most apparent avenue for further research is the development of second-order variants of other theories of definite descriptions.
- This approach could be further generalized to examine higher-order theories of definite descriptions.
- Another route is to consider the modifications of Russellian theory of DD with a non-classical foundation.
- One could try to develop its second- or higher-order version.
- Rather than adopting alternative theories, one can consider remaining within $\mathbf{RL^2}$ and conducting additional investigation: for example, one could try to find a constructive proof of cut admissibility for this logic, building upon the existing proof for **RL** as presented in (Indrzejczak, Kürbis, 2023).

Thank you for attention!

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