

Deductive Semantics

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1 Gentzen's Remarkable Systematic

Gentzen observed a 'remarkable systematic' in the rules of inference of his calculus *NJ* for intuitionist logic. Here are those for the propositional connectives in his notation:¹

$$\begin{array}{c}
 \frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \wedge \mathcal{B}} \quad \frac{\mathcal{A} \wedge \mathcal{B}}{\mathcal{A}} \quad \frac{\mathcal{A} \wedge \mathcal{B}}{\mathcal{B}} \quad \frac{\mathcal{A}}{\mathcal{A} \vee \mathcal{B}} \quad \frac{\mathcal{B}}{\mathcal{A} \vee \mathcal{B}} \quad \frac{\mathcal{A} \vee \mathcal{B} \quad \begin{array}{c} [\mathcal{A}] \\ \vdots \\ \mathcal{C} \end{array} \quad \begin{array}{c} [\mathcal{B}] \\ \vdots \\ \mathcal{C} \end{array}}{\mathcal{C}} \\
 \frac{\begin{array}{c} [\mathcal{A}] \\ \vdots \\ \mathcal{B} \end{array}}{\mathcal{A} \supset \mathcal{B}} \quad \frac{\mathcal{A} \supset \mathcal{B} \quad \mathcal{A}}{\mathcal{B}} \quad \frac{\begin{array}{c} [\mathcal{A}] \\ \vdots \\ \perp \end{array}}{\neg \mathcal{A}} \quad \frac{\neg \mathcal{A} \quad \mathcal{A}}{\perp} \quad \frac{\perp}{\mathcal{D}}
 \end{array}$$

Each of the 'logical symbols' $\wedge, \vee, \supset, \neg$ is governed by two kinds of rules: one for introducing it as the 'outermost symbol' (main connective) of a formula, and one that eliminates it:

The introductions constitute, so to speak, the "definitions" of the symbols concerned, and the eliminations are in the end only consequences thereof, which could roughly be expressed thus: In the elimination of a symbol, the formula in question, whose outermost symbol it concerns, may only "be used as that which it means on the basis of the introduction of this symbol". An example may elucidate what is meant: the formula $\mathcal{A} \supset \mathcal{B}$ can be introduced if there is a deduction of $\mathcal{B}$ from the assumption formula $\mathcal{A}$. Should one now want to use it again in the elimination of the symbol $\supset$ [...], then this can be done precisely by deducing $\mathcal{B}$ immediately from a proved $\mathcal{A}$, because $\mathcal{A} \supset \mathcal{B}$ records the existence of a deduction of $\mathcal{B}$ from $\mathcal{A}$. Note that in this there is no need to take into account an "intuitive sense" of $\supset$.

Gentzen continues: 'By making these thoughts precise it should be possible to show on the basis of certain requirements that the B-inferences [elimination rules] are unique functions of the related E-inferences [introduction rules]'. (Quotations from Gentzen: *Inv. into Nat. Ded. I*, p.189)

2 Dummett, Prawitz, Harmony, Stability and Normalisation

According to Dummett, the meanings of sentences are determined by the grounds and consequences of their assertion. These grounds and consequences must be *in harmony*: we should not be able to infer more consequences from a sentence than are justified by the grounds for asserting it. The converse should hold, too: we should be able to infer all the consequences from a sentence that are justified by its grounds. Dummett calls both conditions together *stability*.

When this is applied to the connectives, Dummett speaks of determining the elimination rules for a connective $*$ from its introduction rules (cf. Gentzen's function), and conversely, introduction rules from elimination rules. When stability obtains, both procedures lead to the same result. Gentzen takes the introduction rules to define the meanings of the connectives. We could also take the elimination rules to do so. When stability obtains, it does not matter which rules we choose.

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Although suggestive, it is difficult to state precisely what harmony and stability are. The following is a rough and ready approximation familiar from the literature:

Stability. All and only those consequences that are justified by the grounds for the deduction of a sentence with $*$ as main operator as specified by the introduction rules for $*$ are consequences of $A * B$ as specified by the elimination rules for $*$.

Some such principle is found in the work of Dummett, Prawitz, Negri and von Plato, etc..

Genzen's observation provided the foundation of Prawitz's work on normalisation:

Observe that an elimination rule is, in a sense, the inverse of the corresponding introduction rule: by an application of an elimination rule one essentially only restores what had already been established if the major premise of the application was inferred by an application of an introduction rule. This relationship between the introduction and elimination rules is roughly expressed by the following principle, which I shall call the *inversion principle*:

Let α be an application of an elimination rule that has B as a consequence. Then, deductions that satisfy the sufficient condition [given by the introduction rules for the logical constants] for deriving the major premise of α , when combined with deductions of the minor premises of α (if any), already "contain" a deduction of B ; the deduction of B is thus obtainable directly from the given deductions without the addition of α .

[...] The inversion principle says in effect that nothing is "gained" by inferring a formula through introduction for use as a major premiss in an elimination. The principle thus suggests the following *inversion theorem*:

If $\Gamma \vdash A$, then there is a deduction of A from Γ in which no formula occurrence is both the consequent of an application of an I-rule and major premiss of an application of an E-rule. (Prawitz: Nat. Ded. 33f)

There is some consensus that, when applied to the connectives, harmony is captured by the relation between introduction and elimination rules that Prawitz describes. It is spelled out formally by a transformation on deductions: a normalisation theorem establishes on the basis of reduction procedures that maximal formulas, those that are the conclusion of an introduction rule and the major premises of an elimination rule, can be removed from deductions.

Whether there is a suitable formal notion that captures the converse of harmony, and thus stability, is a matter of debate.

3 The Aim of this Talk

I shall take for granted that some such account of the meanings of the connectives makes sense (this may be a big 'if', but there are a sufficient number of people who think it does to make my project worth while): the meanings of the logical connectives are defined completely by their use in deductive arguments; this use is codified in a system of natural deduction by introduction and elimination rules for the connectives; and to define the meaning of a connective completely, certain conditions must be placed on the rules governing it. First of all, they must be in harmony and permit normalisation.

My question is: which further conditions should we impose on rules of inference?

I think that almost all of what I'm going to say is already implicit in the literature (there will be at least two controversial claims, though). But I know of no place where these conditions are all spelled out explicitly, systematically and comprehensively. So this is that I'm planning to do.

4 A General Remark on the Form of Rules

For rules of inference to define the meaning of a connective $*$, they must specify the grounds and consequences of $A * B$ in terms of something that is independent of the meaning of $*$. Neither grounds nor consequences must refer to $*$. Grounds and consequences must be given in such a way that a speaker who does not yet understand it can learn the meaning of $*$ by being taught the rules of inference governing $*$. Thus the grounds must be given in such a way, that a speaker who does not yet understand $*$ can recognise that the conditions for an application of $*$ introduction obtain and then be taught the meaning

of $*$ by being taught to apply the introduction rule in these circumstances. Similarly for elimination rules: the consequences of $A * B$ must be specified independently of $A * B$. A speaker who does not yet understand $*$ can then be taught the meaning of $*$ by being taught that under conditions he is already in a position to recognise as obtaining certain consequences follow from $A * B$.

Consequently, there should be no discharged assumptions above major premises of elimination rules. For this would presuppose that $A * B$ is the conclusion of a deduction, that is, that it has meaning. A speaker would have to be able to conclude $A * B$ from premises, and this requires understanding it.

1. In introduction rules, $*$ must only occur in the conclusion.
2. In elimination rules, $*$ must only occur in the major premise.

5 Uniqueness

This criterion was put forward by Belnap: if the meaning of $*$ is supposed to be defined completely by its use in deductive arguments, then if the rules for $*'$ are got from the rules for $*$ by replacing the latter by the former, they should be unique in the following sense: if $*$ is replaced by $*'$, and conversely, all inferential relations remain unaffected. Under certain assumption concerning deducibility this amounts to the interderivability of $A * B$ and $A *' B$: $A * B \vdash A *' B$ and $A *' B \vdash A * B$.

This condition strikes me as evidently necessary. If the meanings of $*$ and $*'$ are determined completely by their use in deductive arguments, this use should be identical. If either $A * B \not\vdash A *' B$ or $A *' B \not\vdash A * B$, then their use is different, and hence their meanings are not identical. And so there would have to be something else to the meanings of $*$ and $*'$, and so, as their rules are the same, these rules do not define their meanings completely.

An example of a non-unique connective: in minimal logic $\wedge$ has no rules. (But its meaning is not supposed to be given by rules: it is supposed to be an independently given unacceptable sentence.)

6 Non-Circularity

Like any definitions, the definitions of the meanings of connectives in terms of rules of inference must be formulated in terms of something independent and they must not be circular. The present account assumes that there are meaningful sentences that can be premises and discharged assumptions of rules. In the most basic case, these are the atomic sentences. Their meanings are not given by proof theory.

1. Suppose we replaced the introduction rule for $\supset$ by the following:

$$\frac{A \supset (C \supset B) \quad A \supset C}{A \supset B}$$

This rule does not specify the grounds of inferring $A \supset B$ independently of $\supset$. As the grounds already contain $\supset$, they presuppose that $\supset$ is meaningful. This rule does not define the meaning of $\supset$.

2. Let $*$ be governed by the sole rule:

$$\frac{*A}{*A}$$

Count it as its introduction and elimination rule. The grounds and consequences of $*A$ are balanced. But I doubt that they impart any meaning onto $*$. $*$ is appealed to in the grounds for its assertion.

3. Let the two connectives $\#$ and $*$ be governed by the following rules:

$$\begin{array}{ll} *I: \frac{\#A \quad \#B}{A * B} & \begin{array}{c} [\#A]^i, [\#B]^i \\ \Pi \\ C \\ *E: \frac{A * B}{C} \end{array} \\ \#I: \frac{A * B}{\#A} \quad \frac{A * B}{\#B} & \begin{array}{c} [A * B]^i \\ \Pi \\ C \\ \#E: \frac{\#A \quad \#B}{C} \end{array} \end{array}$$

Grounds and consequences are balanced. But I do not think that we have defined the meanings of # and *. The attempt is circular. As an explanation of the meaning of #, its rules must assume that * already has meaning, and as an explanation of the meaning of *, its rules must assume that # already has meaning.

4. Read presents a connective • with the following rules:

$$\bullet I : \frac{\neg \bullet}{\bullet} \quad \bullet E : \frac{\frac{\frac{\neg \bullet}{\bullet} i}{\Pi} C}{\bullet} i$$

The grounds and consequences of • are balanced. Yet on my view the rules do not define the meaning of •: the premise of •I appeals to • and so does the discharged assumption of •E. The definition is circular.

5. The definition of classical negation requires the rule: From $\Gamma, \neg \mathcal{A} \vdash \perp$, infer $\Gamma \vdash \mathcal{A}$. This is circular.

There may be connectives that are definable only after others have been defined (e.g. a definite description operator can only be defined in terms of identity). So there may be connectives the rules of which appeal to other connectives. Then these other connectives need to have been defined first. So there must be no circular dependencies of meaning, where the rules for a connective c_1 appeal to a connective c_2 , which appeals to a connective c_3 etc. until we reach c_1 again.

7 Singularity

Došen proposed the following condition: The connective should occur only once in each rule by which it is governed. His reason is that if * occurred more than once, the rules do not specify the meaning of *, but of multiple occurrences of *.

I find this plausible, but if I had to give up one of my principles, this would be it. (Došen, too, is a little hesitant and allows that there may be weaker accounts that relax this condition.)

It follows from what had been said earlier that a connective should occur only exactly once in the conclusion of an introduction rule and only exactly once in the major premise of an elimination rule.

8 The Subformula Principle

There must be a significant relation between the premises and discharged hypotheses and the conclusions or major premises of rules of inference. They are not arbitrary relative to each other. Consider an example where there is no relation between premise and conclusion:

$$\frac{A}{B}$$

Clearly this rule is not legitimate: the grounds for B , i.e. A , stand in no relation to it.

Contrast with the rules of intuitionist logic: In every introduction rule all the premises are proper subformulas of the conclusion, and taking all the introduction rules, their premises together exhaust the proper subformulas of their conclusion.

The grounds for deducing a formula must somehow be ‘preserved’ in it for it to be possible to retrieve them again in drawing consequences from it. As Gentzen said, formulas ‘record’ the grounds sufficient for applying an introduction rule, and as Prawitz said, the elimination rules restore those grounds. I propose that this is achieved through a subformula principle that rules must fulfil.

For introduction rules the following should be the case: All premises and discharged hypotheses of an introduction rule are proper subformulas of its conclusion, and every proper subformula of the conclusion is a premise or discharged hypothesis of some introduction rule for the connective.

It is more difficult to state the corresponding principle for elimination rules, because of the difference in form of those for disjunction and for the other connectives. Call the conclusions of side-deductions that also provide the conclusion of an elimination rule its arbitrary premises, any other premises not containing the constant its minor premises. There are two options:

(a) All minor premises and the conclusion of an elimination rule are proper subformulas of the major premise, and every proper subformula of the major premise is a minor premise or conclusion of some elimination rule for the connective.

(b) All minor premises and all discharged assumptions of an elimination rule are proper subformulas of the major premise, and all proper subformulas of the major premise are minor premises or discharged assumptions of some elimination rule for the connective.