# A Paradefinite Version of Russellian Theory of Definite Descriptions

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# Introduction

- In this talk, we introduce a paradefinite (both paraconsistent and paracomplete) version of Russell's logic of definite descriptions.
- Paraconsistency preserves nontrivial reasoning in the presence of inconsistent information, while paracompleteness allows one to reason with gaps; bringing these two features to definite descriptions is crucial for handling existence/uniqueness constraints under partial and conflicting data.
- We present a bivalent semantics with a paradefinite satisfaction relation that admits an explicit four-valued reading, based on Belnap–Dunn's logic of First Degree Entailment.
- We introduce a sound, complete, and cut-free sequent calculus amenable to proof-search.

- We use the term ‘paradefinite logic’ (‘beyond the definite’), following Arieli and Avron [1], who use it for logics that are both paraconsistent and paracomplete. Such logics are also referred to as “non-alethic” by da Costa and “paranormal” by Béziau. Following da Costa [2], Arieli and Avron [1, p. 1087] characterize
- paraconsistency as “the ability to properly handle contradictory data by rejecting the principle of explosion, according to which any proposition can be inferred from an inconsistent set of assumptions” and
- paracompleteness as “the ability to properly handle incomplete data by rejecting the law of excluded middle, according to which for any proposition, either that proposition is ‘true’ (i.e., known) or its negation is ‘true’.”


Arieli, O., Avron, A.: Four-valued paradefinite logics. Studia Logica **105**, 1087–1122 (2017)

da Costa, N.C.A.: On the theory of inconsistent formal systems. Notre Dame J. Formal Log. **15**, 497–510 (1974)

- As Arieli and Avron write [1, p. 1088], four truth values suffice for reasoning with both incompleteness and inconsistency, so four-valued semantics offers a minimal and natural basis for paradefinite reasoning.
- This perspective goes back to Belnap's computer (inspired by Dunn's studies) which is an abstract reasoning engine processing inputs from multiple, potentially inconsistent or incomplete sources of information. In this paper, we adopt that paradigm but, for technical economy, we work with a bivalent semantics with a paradefinite satisfaction of formulas equivalent to Belnap's four-valued approach.


Belnap, N.D.: How a computer should think. In: Rule, G. (ed.) *Contemporary Aspects of Philosophy*, pp. 30–56. Oriel Press, Newcastle (1977)

Dunn, J.M.: Intuitive semantics for first-degree entailment and coupled trees. Philos. Stud. **29**, 149–168 (1976)

- The other line of our research is definite descriptions, which are typically first-order expressions constructed with the term-forming operator $\iota$, denoting (when it exists) the unique object satisfying a given property.
- Definite descriptions may be proper or improper. Proper ones satisfy the existence and uniqueness conditions (e.g., ‘the sum of 21 and 9’), whereas improper ones do not (e.g., ‘the capital of South Africa’, no uniqueness, since Cape Town, Pretoria, and Bloemfontein serve as capitals; ‘Mozart’s concerto for cello and orchestra’, no such work exists).

Definite descriptions in paraconsistent setting have been studied, notably by Priest [2], who treats definite (and indefinite) descriptions via choice functions over the semantics of a three-valued paraconsistent logic of paradox **LP** [3], allowing non-unique and even non-existent denotation in intensional contexts. By contrast, we work in an extensional setting and stay closer to the original Russellian approach. Also, unlike Priest, in addition to semantics, we develop a proof-theoretic account.


- Priest, G.: The logic of paradox. J. Philos. Logic **8**, 219–241 (1979)
- Priest, G.: *Towards Non-Being: The Logic and Metaphysics of Intentionality*, 2nd edn. Oxford University Press, Oxford (2016)

In [2], definite descriptions, formalized via Kürbis' [1] binary quantifier $Ix[F, G]$, were investigated in first-order version of Nelson's paraconsistent logic **N4** and negative-free variant. Proof-theoretically, [2] characterizes DDs via a natural-deduction system, but neither a normalization theorem nor a subformula property is established. Semantically, it is based on the Kripke semantics and the treatment of identity is comparatively involved. Our approach is extensional and yields simpler truth and falsity clauses for (in)equality and descriptions. Proof-theoretically, we provide a cut-free sequent calculus with cut admissibility and a negation subformula property, supplying the proof-search backbone that the natural-deduction approach in [2] does not deliver.


- Kürbis, N.: A binary quantifier for definite descriptions in intuitionist negative free logic: natural deduction and normalization. Bull. Sect. Logic **48**(2), 81–97 (2019)
- Petrukhin, Y.: A binary quantifier for definite descriptions in Nelsonian free logic. In: Indrzejczak, A., Zawidzki, M. (eds.) *Proceedings of the Eleventh International Conference on Non-Classical Logics. Theory and Applications*. EPTCS, vol. 415, pp. 5–15 (2024)

There are multiple theories of definite descriptions, prompting the questions of

(i) which framework is generally most appropriate and
(ii) which offers the best foundation for a paradefinite treatment.

We adopt the Russellian approach: notwithstanding familiar controversies, it is the standard point of reference across the literature and textbooks and is widely employed by formal logicians. Three characteristic features of the Russellian theory are that

(a) definite descriptions are considered only together with some predicate,
(b) even when a description is improper, such sentences admit clear truth and falsity conditions, and
(c) descriptions are uniformly eliminable from the language via an equivalent first-order formula (see below).

Russell, B.: On denoting. Mind **14**, 479–493 (1905)

Whitehead, A.N., Russell, B.: Principia Mathematica, vol. I. Cambridge University Press, Cambridge (1910)

- As a starting point we adopt the logic **RL**, originally introduced by Indrzejczak and Zawidzki (via a semantics and analytic tableaux) and later reformulated as a cut-free sequent calculus by Indrzejczak and Kürbis. In what follows, we refer to the latter presentation.
- A characteristic feature of this treatment of Russellian definite descriptions is the use of $\lambda$-abstraction as an explicit scope marker. Thus, the Russellian analysis is rendered as

$$(\lambda x\psi)\iota y\varphi \ := \ \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi),$$

instead of the more standard substitutional form

$$\psi(x/\iota y\varphi) \ := \ \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi).$$

Indrzejczak, A., Zawidzki, M.: When Iota meets Lambda. Synthese **201**(2), 1–33 (2023)

Indrzejczak, A., Kürbis, N.: A Cut-Free, Sound and Complete Russellian Theory of Definite Descriptions. In: Ramanayake, R., Urban, J. (eds.) Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2023. LNCS, vol. 14278, pp.

- The substitutional approach is often restricted to *atomic* $\psi$, to avoid scope-related pathologies; in particular, $\neg\psi(x/\iota y\varphi)$ is ambiguous (does the negation take wide scope over the whole formula, or only over the predicate $\psi$?). The $\lambda$-formulation removes this ambiguity by making scope explicit.
- A further advantage is that it supports a relatively simple cut-free sequent calculus.

We introduce a paradefinite extension of **RL**, which we call **PRL**. As often happens in paraconsistent and paracomplete logics, one must handle the negation of each kind of formula separately. For the negation of definite descriptions we stipulate the following clause:

$$\neg(\lambda x \psi)\iota y \varphi \;:=\; \forall x(\exists y \neg(\varphi \leftrightarrow y = x) \vee \neg\psi).$$

# Language and semantics

- The language $\mathscr{L}$ of **RL** and **PRL** is a standard first-order language with the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, with identity, $\lambda$, and $\iota$, and without function symbols.
- We write $\neq$ for the negation of $=$.
- $\mathscr{L}$ contains
  - the set $VAR = \{x, y, z, x_1, \ldots\}$ of bound variables,
  - the set $PAR = \{a, b, c, a_1, \ldots\}$ of free variables (parameters),
  - the set $CON = \{k, k_1, k_2, \ldots\}$ of constants.
- The distinction between $VAR$ and $PAR$ matters for proof theory but not for semantics.
- The basic terms of the language consist of variables, parameters, and constants.

- Following Indrzejczak and Kürbis, we call
  - $(\lambda x \varphi)$ a predicate abstract and $\iota x \varphi$ a quasi-term, if $\varphi$ is a formula;
  - $\varphi t$ a formula (lambda atom), if $\varphi$ is a predicate abstract and $t$ is a term or a quasi-term.
- As usual, if $P^n$ is an $n$-ary predicate symbol (including $=$) and $t_1, \ldots, t_n$ are terms, then $P^n(t_1, \ldots, t_n)$ is a formula (atomic formula).
- In **PRL**, $\neg P^n(t_1, \ldots, t_n)$ is also an atomic formula.
- If $\varphi$ and $\psi$ are formulas, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, $\forall x \varphi$, and $\exists x \varphi$ are formulas.
- We write $\varphi^x_t$ for the result of replacing $x$ by $t$ in $\varphi$. When $t$ is a variable $y$, we assume that $y$ is free for $x$ in $\varphi$, which means that substitution does not cause any formerly free occurrence of $y$ to become bound within $\varphi$.

A *model* is a pair $M = \langle D, I \rangle$, where for each $n$-argument predicate $P^n$, $I(P^n) \subseteq D^n$, and for each constant $k$, $I(k) \in D$. An *assignment* $v$ is a function $v : VAR \cup PAR \mapsto D$. An *x-variant* $v'$ of $v$ agrees with $v$ on all arguments, except possibly $x$. We write $v_o^x$ to denote the $x$-variant of $v$ by $v_o^x(x) = o$.

The notion of a *satisfaction* of a formula $\varphi$ *with* $v$, in symbols $M, v \models \varphi$, is defined as follows, where $t$ is a term:

$$
\begin{aligned}
M, v \models P^n(t_1, \ldots, t_n) &\text{ iff } \langle v(t_1), \ldots, v(t_n)\rangle \in I(P^n) \\
M, v \models t_1 = t_2 &\text{ iff } v(t_1) = v(t_2) \\
M, v \models (\lambda x\psi)t &\text{ iff } M, v^x_o \models \psi, \text{ where } o = v(t) \\
M, v \models (\lambda x\psi)\iota y\varphi &\text{ iff there is an } o \in D \text{ such that } M, v^x_o \models \psi, \\
&\quad M, v^x_o \models \varphi^y_x, \text{ and for any } y\text{-variant } v' \text{ of } v^x_o, \\
&\quad \text{if } M, v' \models \varphi, \text{ then } v'(y) = o \\
M, v \models \neg\varphi &\text{ iff } M, v \not\models \varphi \\
M, v \models \varphi \wedge \psi &\text{ iff } M, v \models \varphi \text{ and } M, v \models \psi \\
M, v \models \varphi \vee \psi &\text{ iff } M, v \models \varphi \text{ or } M, v \models \psi \\
M, v \models \varphi \to \psi &\text{ iff } M, v \models \varphi \text{ implies } M, v \models \psi \\
M, v \models \varphi \leftrightarrow \psi &\text{ iff } M, v \models \varphi \text{ iff } M, v \models \psi \\
M, v \models \forall x\varphi &\text{ iff } M, v^x_o \models \varphi, \text{ for all } o \in D \\
M, v \models \exists x\varphi &\text{ iff } M, v^x_o \models \varphi, \text{ for some } o \in D.
\end{aligned}
$$

A *paradefinite model* is a pair $M = \langle D, I \rangle$, where for each $n$-argument predicate $P^n$ and its negation, we have $I(P^n) \subseteq D^n$ and $I(\neg P^n) \subseteq D^n$; and for each constant $k$, $I(k) \in D$. An *assignment* and an *x-variant* are understood as in the definition of a model.

The notion of a *paradefinite satisfaction* of a formula $\varphi$ *with* $v$, in symbols $M, v \models^{\mathfrak{p}} \varphi$, is defined as follows: all the conditions from the definition of a satisfaction, except the one for negation, hold together with the following ones, where $t$ is a term:

$$
\begin{aligned}
M, v \models^{\mathfrak{p}} \neg P^n(t_1, \ldots, t_n) &\text{ iff } \langle v(t_1), \ldots, v(t_n) \rangle \in I(\neg P^n) \\
M, v \models^{\mathfrak{p}} t_1 \neq t_2 &\text{ iff } v(t_1) \neq v(t_2) \\
M, v \models^{\mathfrak{p}} \neg(\lambda x \psi)t &\text{ iff } M, v^x_o \models^{\mathfrak{p}} \neg\psi, \text{ where } o = v(t) \\
M, v \models^{\mathfrak{p}} \neg(\lambda x \psi)\iota y \varphi &\text{ iff for all } o \in D \text{ it holds that } M, v^x_o \models^{\mathfrak{p}} \neg\psi, \text{or} \\
&\quad M, v^x_o \models^{\mathfrak{p}} \neg\varphi^y_x, \text{ or there is an } y\text{-variant } v' \text{ of } v^x_o, \\
&\quad \text{such that } M, v' \not\models^{\mathfrak{p}} \neg\varphi \text{ and } v'(y) \neq o \\
M, v \models^{\mathfrak{p}} \neg\neg\varphi &\text{ iff } M, v \models^{\mathfrak{p}} \varphi \\
M, v \models^{\mathfrak{p}} \neg(\varphi \wedge \psi) &\text{ iff } M, v \models^{\mathfrak{p}} \neg\varphi \text{ or } M, v \models^{\mathfrak{p}} \neg\psi \\
M, v \models^{\mathfrak{p}} \neg(\varphi \vee \psi) &\text{ iff } M, v \models^{\mathfrak{p}} \neg\varphi \text{ and } M, v \models^{\mathfrak{p}} \neg\psi \\
M, v \models^{\mathfrak{p}} \neg(\varphi \rightarrow \psi) &\text{ iff } M, v \models^{\mathfrak{p}} \varphi \text{ and } M, v \models^{\mathfrak{p}} \neg\psi \\
M, v \models^{\mathfrak{p}} \neg(\varphi \leftrightarrow \psi) &\text{ iff either } (M, v \models^{\mathfrak{p}} \varphi \text{ and } M, v \models^{\mathfrak{p}} \neg\psi) \\
&\quad \text{or } (M, v \models^{\mathfrak{p}} \neg\varphi \text{ and } M, v \models^{\mathfrak{p}} \psi) \\
M, v \models^{\mathfrak{p}} \neg\forall x \varphi &\text{ iff } M, v^x_o \models^{\mathfrak{p}} \neg\varphi, \text{ for some } o \in D \\
M, v \models^{\mathfrak{p}} \neg\exists x \varphi &\text{ iff } M, v^x_o \models^{\mathfrak{p}} \neg\varphi, \text{ for all } o \in D.
\end{aligned}
$$

- The notions of a satisfiable formula and a valid formula in **RL** and **PRL** are defined in a standard way.
- The consequence relation in **RL** is defined as follows, for all finite multisets of formulas $\Gamma$ and $\Delta$:
- $\Gamma \models_{\mathbf{RL}} \Delta$ iff in every model $M$ and every assignment $v$, if $M, v \models \psi$, for all $\psi \in \Gamma$, then $M, v \models \chi$, for some $\chi \in \Delta$.
- In the case of **PRL**, the definition is similar: simply consider paradefinite models and replace $\models$ with $\models^{\mathfrak{p}}$.
- A sequent is an ordered pair written as $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite multisets of formulas.
- Let $\mathbf{L} \in \{\mathbf{RL}, \mathbf{PRL}\}$. We write $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$ iff $\Gamma \models_{\mathbf{L}} \Delta$.
- If $\mathcal{S}$ is a set of sequents and $S$ is a sequent, we write $\mathcal{S} \models_{\mathbf{L}} S$ iff $\models_{\mathbf{L}} T$, for all $T \in \mathcal{S}$, implies $\models_{\mathbf{L}} S$.

- The semantics described above yields a paradefinite logic, i.e., both paraconsistent and paracomplete.
- By imposing constraints on paradefinite satisfaction, one can obtain a logic that is solely paraconsistent or solely paracomplete.
- If we require that, for each formula $\varphi$, it is not the case that both $M, v \models^{\mathfrak{p}} \varphi$ and $M, v \models^{\mathfrak{p}} \neg\varphi$, then the result is a paracomplete, but not paraconsistent logic. Let us call it $\mathbf{PRL}^n$.
- If we require that, for each formula $\varphi$, it is not the case that both $M, v \not\models^{\mathfrak{p}} \varphi$ and $M, v \not\models^{\mathfrak{p}} \neg\varphi$, then we get a paraconsistent but not paracomplete logic. Let us call it $\mathbf{PRL}^b$.
- Imposing both constraints simultaneously collapses the system to **RL**.

- The semantics described above is formally two-valued. However, for each formula $\varphi$, there are truth and falsity condition both for $\varphi$ and $\neg\varphi$. Thus, there are four possibilities, and the semantics is de facto four-valued. This can be made explicit by considering the matrix $\mathfrak{M}_4 = \langle V_4 = \{t, b, n, f\}, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, D = \{t, b\}\rangle$.
- The truth values are interpreted in Belnap's style, using the metaphor of a computer receiving potentially incomplete and inconsistent information from unreliable sources:
  - $t$ stands for 'true' (the computer has been told that $\varphi$ is true),
  - $b$ for 'both true and false' (one source reports $\varphi$ as true, another as false),
  - $n$ for 'neither true nor false' (no information is available),
  - $f$ stands for 'false' (the computer has been told that $\varphi$ is false).
- The designated values are $t$ and $b$, so the consequent relation is defined via their preservation.

- The truth tables below for ¬, ∧, and ∨ are due to Belnap. This fragment of $\mathfrak{M}_4$ defines Belnap-Dunn's logic of First Degree Entailment **FDE**.
- The truth table for → is due to Avron, yielding an implicative extension of **FDE** which he calls **Be**.

| $A$ | $\neg$ |
|---|---|
| $t$ | $f$ |
| $b$ | $b$ |
| $n$ | $n$ |
| $f$ | $t$ |

| $\vee$ | $t$ | $b$ | $n$ | $f$ |
|---|---|---|---|---|
| $t$ | $t$ | $t$ | $t$ | $t$ |
| $b$ | $t$ | $b$ | $t$ | $b$ |
| $n$ | $t$ | $t$ | $n$ | $n$ |
| $f$ | $t$ | $b$ | $n$ | $f$ |

| $\wedge$ | $t$ | $b$ | $n$ | $f$ |
|---|---|---|---|---|
| $t$ | $t$ | $b$ | $n$ | $f$ |
| $b$ | $b$ | $b$ | $f$ | $f$ |
| $n$ | $n$ | $f$ | $n$ | $f$ |
| $f$ | $f$ | $f$ | $f$ | $f$ |

| $\rightarrow$ | $t$ | $b$ | $n$ | $f$ |
|---|---|---|---|---|
| $t$ | $t$ | $b$ | $n$ | $f$ |
| $b$ | $t$ | $b$ | $n$ | $f$ |
| $n$ | $t$ | $t$ | $t$ | $t$ |
| $f$ | $t$ | $t$ | $t$ | $t$ |

| $\leftrightarrow$ | $t$ | $b$ | $n$ | $f$ |
|---|---|---|---|---|
| $t$ | $t$ | $b$ | $n$ | $f$ |
| $b$ | $b$ | $b$ | $n$ | $f$ |
| $n$ | $n$ | $n$ | $t$ | $t$ |
| $f$ | $f$ | $f$ | $t$ | $t$ |


Avron, A.: Natural 3-valued logics—characterization and proof theory. J. Symbolic Log. **56**(1), 276–294 (1991)

Belnap, N.D.: A useful four-valued logic. In: Dunn, J.M., Epstein, G. (eds.) *Modern Uses of Multiple-Valued Logic*, pp.

- Quantifiers can be defined as greatest lower bound and least upper bound with respect to the following partial order on $V_4$: $f \leq n \leq t$ and $f \leq b \leq t$, where $n$ and $b$ are incomparable. For details on quantifiers in Belnap's four-valued logic, see [4].
- One may also restrict this semantics to three values: with the set $V_3^b = \{t, b, f\}$ one gets Asenjo's [1] and Priest's [3, 4] paraconsistent logic of paradox **LP**;
- with the set $V_3^n = \{t, n, f\}$ one gets the paracomplete strong Kleene logic $\mathbf{K_3}$ [2].


Asenjo, F.G.: A calculus of antinomies. Notre Dame J. Formal Log. **7**, 103–105 (1966)

Kleene, S.C.: On a notation for ordinal numbers. J. Symbolic Log. **3**(1), 150–155 (1938)

Priest, G.: The logic of paradox. J. Philos. Logic **8**, 219–241 (1979)

Priest, G.: Paraconsistent logic. In: Gabbay, D.M., Guenthner, F. (eds.) *Handbook of Philosophical Logic*, vol. 6. Springer, Dordrecht (2002)

# Sequent calculi

On the next slides, we present Indrzejczak and Kürbis' sequent calculus for **RL**; it is an extension of Troelstra and Schwichtenberg's calculus G1c by rules for $=$, $\lambda$, and $\iota$. The rules for $=$ go back to Negri and von Plato.


- Indrzejczak, A., Kürbis, N.: A cut-free, sound and complete Russellian theory of definite descriptions. In: Ramanayake, R., Urban, J. (eds.) *Automated Reasoning with Analytic Tableaux and Related Methods*. TABLEAUX 2023. LNCS, vol. 14278, pp. 112–130. Springer, Cham (2023)
- Negri, S., von Plato, J.: *Structural Proof Theory*. Cambridge University Press, Cambridge (2001)
- Troelstra, A.S., Schwichtenberg, H.: *Basic Proof Theory*. Oxford University Press, Oxford (1996)

## Sequent calculi. Kürbis and Indrzejczak's calculus for **RL**

$$\text{(Cut)}\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad \text{(AX)}\ \varphi \Rightarrow \varphi$$

$$\text{(W}{\Rightarrow}\text{)}\ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \text{(}{\Rightarrow}\text{W)}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \qquad \text{(C}{\Rightarrow}\text{)}\ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$\text{(}{\Rightarrow}\text{C)}\ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \qquad (\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\wedge \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$$

$$(\leftrightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi \quad \varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \leftrightarrow \psi, \Gamma \Rightarrow \Delta} \quad (\forall \Rightarrow)\ \frac{\varphi^x_b, \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\leftrightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \leftrightarrow \psi} \quad (\Rightarrow \forall)\ \frac{\Gamma \Rightarrow \Delta, \varphi^x_a}{\Gamma \Rightarrow \Delta, \forall x\varphi}$$

$$(\exists \Rightarrow)\ \frac{\varphi^x_a, \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta} \quad (\Rightarrow \exists)\ \frac{\Gamma \Rightarrow \Delta, \varphi^x_b}{\Gamma \Rightarrow \Delta, \exists x\varphi} \quad (= +)\ \frac{b = b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(= -)\ \frac{\mathscr{A}^x_c, \Gamma \Rightarrow \Delta}{b = c, \mathscr{A}^x_b, \Gamma \Rightarrow \Delta} \quad (\lambda \Rightarrow)\ \frac{\psi^x_b, \Gamma \Rightarrow \Delta}{(\lambda x\psi)b, \Gamma \Rightarrow \Delta} \quad (\Rightarrow \lambda)\ \frac{\Gamma \Rightarrow \Delta, \psi^x_b}{\Gamma \Rightarrow \Delta, (\lambda x\psi)b}$$

where $a$ is a fresh parameter (*Eigenvariable*), not present in $\Gamma, \Delta$ and $\varphi$, whereas $b, c$ are arbitrary parameters. $\mathscr{A}$ in $(= -)$ is an atomic formula.

$$(\iota_1 \Rightarrow)\ \frac{\varphi^y_a, \psi^x_a, \Gamma \Rightarrow \Delta}{(\lambda x \psi)\iota y \varphi, \Gamma \Rightarrow \Delta} \quad (\iota_2 \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi^y_b \quad \Gamma \Rightarrow \Delta, \varphi^y_c \quad b = c, \Gamma \Rightarrow \Delta}{(\lambda x \psi)\iota y \varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \iota)\ \frac{\Gamma \Rightarrow \Delta, \varphi^y_b \quad \Gamma \Rightarrow \Delta, \psi^x_b \quad \varphi^y_a, \Gamma \Rightarrow \Delta, a = b}{\Gamma \Rightarrow \Delta, (\lambda x \psi)\iota y \varphi}$$

where $a$ is a fresh parameter (*Eigenvariable*), not present in $\Gamma, \Delta$ and $\varphi$, whereas $b, c$ are arbitrary parameters.

Let $\mathbf{L} \in \{\mathbf{RL}, \mathbf{PRL}\}$. If $\mathcal{S}$ is a set of sequents and $S$ is a sequent, we write $\mathcal{S} \vdash_{\mathbf{L}} S$ iff there exists a tree whose nodes are sequents such that the leaves are axioms or members of $\mathcal{S}$, the root is $S$, and each node is obtained from its immediate predecessors by applying a calculus rule. If $\mathcal{S}$ is a set of sequents and $S$ is a sequent, then we write $\mathcal{S} \vdash^{cf}_{\mathbf{L}} S$ iff $\mathcal{S} \vdash_{\mathbf{L}} S$ and each cut is on a formula that belongs to $\mathcal{S}$.

**Theorem.**

Sequent calculus for **RL** is sound, complete, and cut-free.

Indrzejczak, A., Kürbis, N.: A Cut-Free, Sound and Complete Russellian Theory of Definite Descriptions. In: Ramanayake, R., Urban, J. (eds.) Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2023, LNCS, vol. 14278, pp. 112–130. Springer, Cham (2023)

A sequent calculus for **PRL** is obtained from the one for **RL** by the replacement of the rules $(\neg\Rightarrow)$ and $(\Rightarrow\neg)$ with the following negated rules (to be continued on the next slide):

$$(\Rightarrow\neg\neg)\ \frac{\Gamma\Rightarrow\Delta,\varphi}{\Gamma\Rightarrow\Delta,\neg\neg\varphi}\qquad(\neg\neg\Rightarrow)\ \frac{\varphi,\Gamma\Rightarrow\Delta}{\neg\neg\varphi,\Gamma\Rightarrow\Delta}$$

$$(\neg\wedge\Rightarrow)\ \frac{\neg\varphi,\Gamma\Rightarrow\Delta\quad\neg\psi,\Gamma\Rightarrow\Delta}{\neg(\varphi\wedge\psi),\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\neg\wedge)\ \frac{\Gamma\Rightarrow\Delta,\neg\varphi,\neg\psi}{\Gamma\Rightarrow\Delta,\neg(\varphi\wedge\psi)}$$

$$(\neg\vee\Rightarrow)\ \frac{\neg\varphi,\neg\psi,\Gamma\Rightarrow\Delta}{\neg(\varphi\vee\psi),\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\neg\vee)\ \frac{\Gamma\Rightarrow\Delta,\neg\varphi\quad\Gamma\Rightarrow\Delta,\neg\psi}{\Gamma\Rightarrow\Delta,\neg(\varphi\vee\psi)}$$

$$(\neg\!\to\Rightarrow)\ \frac{\varphi,\neg\psi,\Gamma\Rightarrow\Delta}{\neg(\varphi\to\psi),\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\neg\!\to)\ \frac{\Gamma\Rightarrow\Delta,\varphi\quad\Gamma\Rightarrow\Delta,\neg\psi}{\Gamma\Rightarrow\Delta,\neg(\varphi\to\psi)}$$

$$(\neg\!\leftrightarrow\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\neg\varphi,\psi\quad\varphi,\neg\psi,\Gamma\Rightarrow\Delta}{\neg(\varphi\leftrightarrow\psi),\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\neg\!\leftrightarrow_1)\ \frac{\Gamma\Rightarrow\Delta,\varphi\quad\Gamma\Rightarrow\Delta,\neg\psi}{\Gamma\Rightarrow\Delta,\neg(\varphi\leftrightarrow\psi)}$$

$$(\Rightarrow\neg\!\leftrightarrow_2)\ \frac{\Gamma\Rightarrow\Delta,\neg\varphi\quad\Gamma\Rightarrow\Delta,\psi}{\Gamma\Rightarrow\Delta,\neg(\varphi\leftrightarrow\psi)}$$

# $a$ is a fresh parameter and $\mathscr{A}$ in $(\neq -)$ is atomic

$$(\Rightarrow \neg\forall)\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi^x_b}{\Gamma \Rightarrow \Delta, \neg\forall x\varphi} \quad (\neg\forall \Rightarrow)\ \frac{\neg\varphi^x_a, \Gamma \Rightarrow \Delta}{\neg\forall x\varphi, \Gamma \Rightarrow \Delta} \quad (\neg\exists \Rightarrow)\ \frac{\neg\varphi^x_b, \Gamma \Rightarrow \Delta}{\neg\exists x\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \neg\exists)\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi^x_a}{\Gamma \Rightarrow \Delta, \neg\exists x\varphi} \quad (\neq -)\ \frac{\Gamma \Rightarrow \Delta, \mathscr{A}^x_b \quad \Gamma \Rightarrow \Delta, \neg\mathscr{A}^x_c}{\Gamma \Rightarrow \Delta, b \neq c}$$

$$(\neq\text{Sym})\ \frac{b \neq c, \Gamma \Rightarrow \Delta}{c \neq b, \Gamma \Rightarrow \Delta} \quad (\neq +)\ \frac{\Gamma \Rightarrow \Delta, b \neq b}{\Gamma \Rightarrow \Delta} \quad (\neg\lambda \Rightarrow)\ \frac{\neg\psi^x_b, \Gamma \Rightarrow \Delta}{\neg(\lambda x\psi)b, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \neg\lambda)\ \frac{\Gamma \Rightarrow \Delta, \neg\psi^x_b}{\Gamma \Rightarrow \Delta, \neg(\lambda x\psi)b} \quad (\Rightarrow \neg\iota_1)\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi^y_a, \neg\psi^x_a}{\Gamma \Rightarrow \Delta, \neg(\lambda x\psi)\iota y\varphi}$$

$$(\Rightarrow \neg\iota_2)\ \frac{\neg\varphi^y_b, \Gamma \Rightarrow \Delta \quad \neg\varphi^y_c, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, b \neq c}{\Gamma \Rightarrow \Delta, \neg(\lambda x\psi)\iota y\varphi}$$

$$(\neg\iota \Rightarrow)\ \frac{\neg\varphi^y_b, \Gamma \Rightarrow \Delta \quad \neg\psi^x_b, \Gamma \Rightarrow \Delta \quad \varphi^y_a, a \neq b, \Gamma \Rightarrow \Delta}{\neg(\lambda x\psi)\iota y\varphi, \Gamma \Rightarrow \Delta}$$

- A sequent calculus for $\mathbf{PRL}^b$ is obtained from the one for **PRL** by adding an additional axiom: $\Rightarrow \varphi, \neg\varphi$.
- A sequent calculus for $\mathbf{PRL}^n$ is obtained from **PRL** by adding an axiom $\varphi, \neg\varphi \Rightarrow$.
- Although we do not analyse these two logics in detail in this section, the results on soundness, completeness, cut admissibility, and the negation subformula property extend to them by the same techniques, requiring only routine modifications.
- The $\{\neg, \rightarrow, \wedge, \vee\}$-fragment of **PRL** coincides with Avron's logic **Be** [1] (also known as **Par** due to Popov [2]). Sequent rules for this fragment are given in [1, 2].


Avron, A.: Natural 3-valued logics—characterization and proof theory. J. Symbolic Log. **56**(1), 276–294 (1991)

Popov, V.M.: Sequent formulations of paraconsistent logical systems (in Russian). In: Smirnov, V. (ed.) *Semantic and Syntactic Investigations of Non-Extensional Logics*, pp. 285–289. Nauka (1989)

- Sequent rules for the $\{\neg, \rightarrow, \wedge, \vee\}$-fragments of $\mathbf{PRL}^b$ and $\mathbf{PRL}^n$ are given in [1].
- The rules for $\leftrightarrow$, $\forall$, and $\exists$ are standard. For $\neg\forall$ and $\neg\exists$, we adopt the usual rules employed in paraconsistent first-order logic; see, e.g., [2].


Avron, A.: Classical Gentzen-type methods in propositional many-valued logics. In: Fitting, M., Orłowska, E. (eds.) *Beyond Two: Theory and Applications of Multiple-Valued Logic*, Studies in Fuzziness and Soft Computing, vol. 114, pp. 117–155. Physica, Heidelberg (2003)

Wansing, H., Weber, Z.: Quantifiers in connexive logic (in general and in particular). Log. J. IGPL (2024). https://doi.org/10.1093/jigpal/jzae115

### Leibniz Lemma

Let $\mathbf{L} \in \{\mathbf{RL}, \mathbf{PRL}\}$ and $n$ be the height of a proof.

1. $\vdash_{\mathbf{L}} \varphi^x_{b_1}, b_1 = b_2 \Rightarrow \varphi^x_{b_2}$, for any formula $\varphi$,
2. if $\vdash^n_{\mathbf{L}} \Gamma \Rightarrow \Delta$, then $\vdash^n_{\mathbf{RL}} \Gamma^{b_1}_{b_2} \Rightarrow \Delta^{b_1}_{b_2}$.

Let $\mathbf{L} \in \{\mathbf{RL}, \mathbf{PRL}\}$.

1. $\vdash_{\mathbf{L}} (\lambda x \psi)\iota y \varphi \Rightarrow \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi)$
2. $\vdash_{\mathbf{L}} \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi) \Rightarrow (\lambda x \psi)\iota y \varphi$
3. $\vdash_{\mathbf{PRL}} \neg(\lambda x \psi)\iota y \varphi \Rightarrow \forall x(\exists y \neg(\varphi \leftrightarrow y = x) \vee \neg\psi)$
4. $\vdash_{\mathbf{PRL}} \forall x(\exists y \neg(\varphi \leftrightarrow y = x) \vee \neg\psi) \Rightarrow \neg(\lambda x \psi)\iota y \varphi$

The proofs of (1) and (2) in **RL** one can find in the paper by Indrzejczak and Kürbis (in **PRL**, the proofs are the same). Let us prove (3) and (4). To make proofs shorter we use multiplicative versions of the rules for $\iota$ and $\leftrightarrow$ and their negations. All the proofs can be rebuilt for additive versions presented above. First of all, let us prove the following sequent which we denote via $\mathfrak{D}$:

$$\dfrac{\dfrac{\varphi^y_{a'} \Rightarrow \varphi^y_{a'} \qquad a' \neq a \Rightarrow a' \neq a}{\varphi^y_{a'}, a' \neq a \Rightarrow \neg(\varphi^{y}{}'_{a} \leftrightarrow a' = a)}\,(\Rightarrow \neg\leftrightarrow)}{\varphi^y_{a'}, a' \neq a \Rightarrow \exists y \neg(\varphi \leftrightarrow y = a)}\,(\Rightarrow \exists)$$

Then we proceed as follows:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \neg\varphi^y_a \Rightarrow \neg\varphi^y_a
      \qquad
      \dfrac{a = a \Rightarrow a = a}{\Rightarrow a = a}\,(= +)
    }{
      \neg\varphi^y_a \Rightarrow \neg(\varphi^y_a \leftrightarrow a = a)
    }\,(\Rightarrow \neg\leftrightarrow)
  }{
    \neg\varphi^y_a \Rightarrow \exists y \neg(\varphi \leftrightarrow y = a)
  }\,(\Rightarrow \exists)
  \qquad
  \neg\psi^x_a \Rightarrow \neg\psi^x_a
  \qquad
  \mathfrak{D}
}{
  \dfrac{
    \dfrac{
      \neg(\lambda x \psi)\iota y \varphi \Rightarrow \exists y \neg(\varphi \leftrightarrow y = a), \neg\psi^x_a
    }{
      \neg(\lambda x \psi)\iota y \varphi \Rightarrow \exists y \neg(\varphi \leftrightarrow y = a) \vee \neg\psi^x_a
    }\,(\Rightarrow \vee)
  }{
    \neg(\lambda x \psi)\iota y \varphi \Rightarrow \forall x(\exists y \neg(\varphi \leftrightarrow y = x) \vee \neg\psi)
  }\,(\Rightarrow \forall)
}\,(\neg\iota \Rightarrow)
$$

Let us prove the following sequent which we denote via $\mathfrak{E}$:

$$\dfrac{\neg\varphi^y_a \Rightarrow \neg\varphi^y_a \qquad \neg\varphi^y_{a'} \Rightarrow \neg\varphi^y_{a'} \qquad \dfrac{a' = a \Rightarrow a' = a, a \neq a'}{\Rightarrow a' = a, a \neq a'}\,(= +)}{\Rightarrow \neg(\lambda x \psi)\iota y \varphi, \neg\varphi^y_a, \neg\varphi^y_{a'}, a' = a}\,(\Rightarrow \neg\iota_2)$$

Then we proceed as follows:

$$
(\Rightarrow\mathrm{C})\,\dfrac{(\Rightarrow\neg\iota_1)\,\dfrac{(\forall\Rightarrow)\,\dfrac{(\vee\Rightarrow)\,\dfrac{(\exists\Rightarrow)\,\dfrac{(\neg\!\leftrightarrow\Rightarrow)\,\dfrac{\mathfrak{E} \qquad \dfrac{\varphi^y_{a'}, a' \neq a \Rightarrow a' \neq a}{\varphi^y_{a'}, a' \neq a \Rightarrow}\,(\neq +)}{\neg(\varphi^y_{a'} \leftrightarrow a' = a) \Rightarrow \neg(\lambda x \psi)\iota y \varphi, \neg\varphi^y_a}}{\exists y \neg(\varphi \leftrightarrow y = a) \Rightarrow \neg(\lambda x \psi)\iota y \varphi, \neg\varphi^y_a} \qquad \neg\psi^x_a \Rightarrow \neg\psi^x_a}{\exists y \neg(\varphi \leftrightarrow y = a) \vee \neg\psi^x_a \Rightarrow \neg(\lambda x \psi)\iota y \varphi, \neg\varphi^y_a, \neg\psi^x_a}}{\forall x(\exists y \neg(\varphi \leftrightarrow y = x) \vee \neg\psi) \Rightarrow \neg(\lambda x \psi)\iota y \varphi, \neg\varphi^y_a, \neg\psi^x_a}}{\forall x(\exists y \neg(\varphi \leftrightarrow y = x) \vee \neg\psi) \Rightarrow \neg(\lambda x \psi)\iota y \varphi, \neg(\lambda x \psi)\iota y \varphi}}{\forall x(\exists y \neg(\varphi \leftrightarrow y = x) \vee \neg\psi) \Rightarrow \neg(\lambda x \psi)\iota y \varphi}
$$

**Theorem.**

Sequent calculus for **PRL** is sound, complete, and cut-free. (Proven by Hintikka-style argument).

# Constructive cut elimination

In [1], a constructive cut-elimination proof for **RL** is given, following the strategy of Metcalfe, Olivetti, and Gabbay [2]. The same strategy would carry over to **PRL**, were it not for a difficulty involving $\neq$. Consider the following deduction:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \mathscr{A}^x_b \qquad \Gamma \Rightarrow \Delta, \neg\mathscr{A}^x_c}{\Gamma \Rightarrow \Delta, b \neq c}\,(\neq -) \qquad \dfrac{c \neq b, \Gamma \Rightarrow \Delta}{b \neq c, \Gamma \Rightarrow \Delta}\,(\neq\text{Sym})}{\Gamma \Rightarrow \Delta}\,(\text{Cut})$$


Indrzejczak, A., Kürbis, N.: A cut-free, sound and complete Russellian theory of definite descriptions. In: Ramanayake, R., Urban, J. (eds.) *Automated Reasoning with Analytic Tableaux and Related Methods*. TABLEAUX 2023. LNCS, vol. 14278, pp. 112–130. Springer, Cham (2023)

Metcalfe, G., Olivetti, N., Gabbay, D.: *Proof Theory for Fuzzy Logics*. Springer (2008)

Since in **PRL**, the negation of predicates are treated as atomic formulas, we can transform the above deduction as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \neg\mathscr{A}^x_c \qquad \dfrac{\Gamma \Rightarrow \Delta, \mathscr{A}^x_b}{\Gamma \Rightarrow \Delta, \neg\neg\mathscr{A}^x_b}\,(\Rightarrow \neg\neg)}{\Gamma \Rightarrow \Delta, c \neq b}\,(\neq -) \qquad c \neq b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\mathrm{Cut})$$

However, this transformation increases both the proof degree and the height of the derivation of the cut's left premise.

Because negation behaves non-classically in our sequent calculus, we adjust the definition of cut-degree used in [1], where it is simply the number of logical constants. In **PRL**, the cut-degree $d(\varphi)$ of the cut-formula $\varphi$ is defined as follows:

$d(P(t_1, \ldots, t_n)) = d(\neg P(t_1, \ldots, t_n)) = d(t_1 = t_2) = d(t_1 \neq t_2) = 0.$

For any non-atomic formula $\varphi$, $d(\varphi)$ is the number of logical constants (connectives, quantifiers, and operators) occurring in $\varphi$. The proof-degree $d(\mathfrak{D})$ of the deduction $\mathfrak{D}$ is the maximal cut-degree in $\mathfrak{D}$.

# Connexive logic.

- In contemporary philosophical logic, a prominent line of work related to paradefinite systems is connexive logic.
- A hallmark of these systems is the treatment of the negation of implication in a way that supports the connexive principles: they validate
- Aristotle's theses, $\neg(\varphi \to \neg\varphi)$ and $\neg(\neg\varphi \to \varphi)$, as well as
- Boethius' theses, $(\varphi \to \psi) \to \neg(\varphi \to \neg\psi)$ and $(\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$.
- Connexive logic is counter-classical, since the foregoing formulas are not classically valid; at the same time it is nontrivial, as some classical tautologies fail (for example, $\neg(\varphi \to \psi) \leftrightarrow (\varphi \wedge \neg\psi)$ does not hold in connexive settings).
- Semantically, a connexive variant of **PRL** can be obtained by revising the truth condition for the negation of implication in the spirit of Wansing's **C**.

- Specifically,

$$M, v \models^{\mathfrak{p}} \neg(\varphi \to \psi) \text{ iff } M, v \models^{\mathfrak{p}} \varphi \text{ implies } M, x \models^{\mathfrak{p}} \neg\psi.$$

- The appropriate sequent rules for $\neg(\varphi \to \psi)$ are as follows [2]:

$$(\neg\!\to\, \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \neg\psi, \Gamma \Rightarrow \Delta}{\neg(\varphi \to \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg\!\to)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \to \psi)}$$

- In [2], the authors additionally propose a second pair of quantifiers, the connexive quantifiers. Building on this, one could develop connexive definite descriptions. We leave this direction to future work.


Wansing, H.: Connexive modal logic. In: Schmidt, R. et al. (eds.) *Advances in Modal Logic*, vol. 5, pp. 367–383. King's College Publications, London (2005)

Wansing, H., Weber, Z.: Quantifiers in connexive logic (in general and in particular). Log. J. IGPL (2024). `https://doi.org/10.1093/jigpal/jzae115`

# Second-order logic

- In [2], a second-order counterpart of **RL** is presented: **RL** extended by second-order quantifiers and second-order definite descriptions of the form $(\lambda X \psi)\iota Y \varphi$, where $X$ and $Y$ are relational variables, refering to unique relations between objects. Second-order identity is also considered: $X = Y =_{df} \forall x_1 \ldots \forall x_n \big(X(x_1, \ldots, x_n) \leftrightarrow Y(x_1, \ldots, x_n)\big)$.
- While full second-order logic is incomplete, it has a complete fragment under Henkin's general models [1]. Accordingly, the second-order extension of **RL** admits both a semantics based on general models and a sound, complete, cut-free sequent calculus extending that of **RL**.
- An analogous second-order extension can be developed for **PRL**.


Henkin, L.: Completeness in the theory of types. J. Symbolic Log. **15**(2), 81–91 (1950)

Petrukhin, Y.: On a second-order version of Russellian theory of definite descriptions. In: Casini, G., Dundua, B., Kutsia, T. (eds.) *Logics in Artificial Intelligence*. JELIA 2025. LNCS, vol. 16094. Springer, Cham (2026)

- The cut admissibility theorem for second- and higher-order logics has remained an unresolved issue in proof theory for an extended duration, referred to as Takeuti's conjecture (Takeuti, 1953). Various scholars, utilizing distinct methodologies, reached a positive resolution: Tait (1966), Prawitz (1968), Takahashi (1967), Girard (1971). Developing a syntactic constructive proof remains an open problem.
- However, we provide a semantic proof of this statement obtained as a consequence of a Hintikka-style completeness proof in the spirit of [1, 2].

Avron, A., Lahav, O.: A simple cut-free system for a paraconsistent logic equivalent to S5. In: Advances in Modal Logic, vol. 12, pp. 29–42. College Publications (2018)

Lahav, O., Avron, A.: A semantic proof of strong cut-admissibility for first-order Gödel logic. Journal of Logic and Computation **23**(1), 59–86 (2013)

# Let us present several examples of the second-order definite descriptions

- The transitive closure of a graph: $\iota R\ \mathsf{TransitiveClosure}(R, G)$.
- The expression

  $$\iota P\ (\mathsf{Path}(P, a, b, G) \wedge \forall P'\,(\mathsf{Path}(P', a, b, G) \to \mathsf{Length}(P) \leq \mathsf{Length}(P')))$$

  represents 'the shortest path relation between two nodes $a$ and $b$ in graph $G$', where $P$ is a path (represented as a relation).
- The connectivity relation in a graph can be formalized as $\iota R\ (\forall x \forall y\ (R(x, y) \leftrightarrow \exists P\ \mathsf{Path}(P, x, y, G)))$.
- Finally, the expression $\iota R\ (\mathsf{TotalOrder}(R) \wedge \forall x \forall y\ (P(x, y) \to R(x, y)))$ denotes the total order extending a given partial order $P$, assuming such an extension is unique.

Let us now describe the logic $\mathbf{PRL^2}$, a second-order generalization of $\mathbf{PRL}$.

- *The language* $\mathscr{L}^2$ *of* $\mathbf{RL^2}$ is a second-order extension of $\mathscr{L}$.
- In addition to individual variables and parameters, we have the sets
- $VAR^2 = \{X, Y, Z, X_1, \ldots\}$ and
- $PAR^2 = \{A, B, C, A_1, \ldots\}$ of $n$-ary relational variables and parameters, respectively (unary ones might be called property variables and parameters).
- As in the first-order case, this distinction is important for proof theory, but might be relaxed in the case of semantics.
- The terms are constants (if added) and individual variables/parameters.
- Notice that relational variables/parameter are not terms.

- Atomic formulas are as follows: $t_1 = t_2$, $t_1 \neq t_2$, $P(t_1, \ldots, t_n)$, $\neg P(t_1, \ldots, t_n)$, $X = Y$, $X \neq Y$, $X(t_1, \ldots, t_n)$, and $\neg X(t_1, \ldots, t_n)$, if $t_1, \ldots, t_n$ are terms, $P$ is an $n$-ary relation symbol, and $X$ and $Y$ are $n$-ary relational variables/parameters.
- The formula $X = Y$ is understood as $\forall x_1 \ldots \forall x_n \big(X(x_1, \ldots, x_n) \leftrightarrow Y(x_1, \ldots, x_n)\big)$.
- The formula $X \neq Y$ is understood as $\exists x_1 \ldots \exists x_n \neg \big(X(x_1, \ldots, x_n) \leftrightarrow Y(x_1, \ldots, x_n)\big)$.
- The formula $t_1 = t_2$ might be defined as $\forall X(X(t_1) \leftrightarrow X(t_2))$.
- In addition to the above described atomic and first-order formulas, we define the following ones:
    - If $\varphi$ is a formula and $X \in VAR^2$, then $\forall X \varphi$ and $\exists X \varphi$ are formulas.
    - If $\varphi$ is a formula, then $(\lambda X \varphi)$ is a *relational* abstract.
    - If $\varphi$ is a formula, then $\iota X \varphi$ is a *pseudo*-term.
    - If $\varphi$ is a relational abstract and $t$ is a pseudo-term, then $\varphi t$ is a formula.
- We write $\mathscr{F}^2$ for the set of all formulas of $\mathscr{L}^2$, $\varphi^X_P$ for the result of replacing $X$ by a predicate symbol $P$ in $\varphi$.

*The first version of the semantics (without a complete calculus).*

- In a paradefinite model $M = \langle D, I \rangle$, an assignment $v$ should be redefined as follows: $v(x) \in D$, for $x \in VAR \cup PAR$, $v(X) \subseteq D^n$, and $v(\neg X) \subseteq D^n$, for $X \in VAR^2 \cup PAR^2$,
- An $X$-*variant* $v'$ of $v$ agrees with $v$ on all arguments, save possibly $X$. We write $v^X_O$ to denote the $X$-variant of $v$ with $v^X_O(X) = O$, where $O \subseteq D^n$.

The definition of the notion of a paradefinite satisfaction of a formula $\varphi$ with $v$ is extended by the following cases, where $t \in VAR \cup PAR$:

$$M, v \models^{\mathfrak{p}} X(t_1, \ldots, t_n) \text{ iff } \langle v(t_1), \ldots, v(t_n) \rangle \in v(X), \text{ if } X \text{ is } n\text{-ary},$$
$$M, v \models^{\mathfrak{p}} \neg X(t_1, \ldots, t_n) \text{ iff } \langle v(t_1), \ldots, v(t_n) \rangle \in v(\neg X), \text{ if } X \text{ is } n\text{-ary},$$
$$M, v \models^{\mathfrak{p}} X = Y \text{ iff } v(X) = v(Y),$$
$$M, v \models^{\mathfrak{p}} X \neq Y \text{ iff } v(X) \neq v(Y),$$
$$M, v \models^{\mathfrak{p}} (\lambda X \psi)\iota Y \varphi \text{ iff there is an } O \subseteq D^n \text{ such that } M, v^X_O \models^{\mathfrak{p}} \psi, M, v^X_O \models^{\mathfrak{p}} \varphi^Y_X, \text{ and for any } Y\text{-variant } v' \text{ of } v^X_O, \text{ if } M, v' \models^{\mathfrak{p}} \varphi, \text{ then } v'(Y) = O$$
$$M, v \models^{\mathfrak{p}} \neg(\lambda X \psi)\iota Y \varphi \text{ iff for all } O \subseteq D^n \text{ it holds that } M, v^X_O \models^{\mathfrak{p}} \neg\psi, \text{or } M, v^X_O \models^{\mathfrak{p}} \neg\varphi^Y_X, \text{ or there is an } Y\text{-variant } v' \text{ of } v^X_O, \text{ such that } M, v' \not\models^{\mathfrak{p}} \neg\varphi \text{ and } v'(Y) \neq O$$
$$M, v \models^{\mathfrak{p}} \forall X \varphi \text{ iff } M, v^X_O \models^{\mathfrak{p}} \varphi, \text{ for all } O \subseteq D^n,$$
$$M, v \models^{\mathfrak{p}} \neg\forall X \varphi \text{ iff } M, v^X_O \models^{\mathfrak{p}} \neg\varphi, \text{ for some } O \subseteq D^n,$$
$$M, v \models^{\mathfrak{p}} \exists X \varphi \text{ iff } M, v^X_O \models^{\mathfrak{p}} \varphi, \text{ for some } O \subseteq D^n,$$
$$M, v \models^{\mathfrak{p}} \neg\exists X \varphi \text{ iff } M, v^X_O \models^{\mathfrak{p}} \neg\varphi, \text{ for all } O \subseteq D^n.$$

- This semantics lacks the completeness theorem.
- In order to obtain this theorem, we need to deal with a fragment of the second-order logic: we should restrict the interpretations of the relational variables/parameters.
- Henkin's concept of a general model will help us with this issue.

Henkin, L.: Completeness in the Theory of Types. J. Symbolic Logic **15**(2), 81–91 (1950)

## *The second version of the semantics (with a complete calculus). The semantics of* $\mathbf{PRL^2}$.

- Although second-order logic is known to be incomplete, there exists a fragment that is complete with respect to general models.
- A *general paradefinite model* is a pair $\mathfrak{M} = \langle M, G \rangle$, where $M = \langle D, I \rangle$ is a paradefinite model and $G$ is a set of subsets, relations (of any arity) on $D$.
- Notice that $v(X) \in G \subseteq \mathcal{P}(D^n)$ and $v(\neg X) \in G \subseteq \mathcal{P}(D^n)$.
- We define the notion of a paradefinite satisfaction of a formula $\varphi$ with $v$ in a general model, symbolically $\mathfrak{M}, v \models^{\mathfrak{p}} \varphi$, for second-order formulas as follows:

$$\mathfrak{M}, v \models^{\mathfrak{p}} X(t_1, \ldots, t_n) \text{ iff } \langle v(t_1), \ldots, v(t_n) \rangle \in v(X), \quad \text{if } X \text{ is } n\text{-ary},$$

$$\mathfrak{M}, v \models^{\mathfrak{p}} \neg X(t_1, \ldots, t_n) \text{ iff } \langle v(t_1), \ldots, v(t_n) \rangle \in v(\neg X), \quad \text{if } X \text{ is } n\text{-ary},$$

$$\mathfrak{M}, v \models^{\mathfrak{p}} X = Y \text{ iff } v(X) = v(Y),$$

$$\mathfrak{M}, v \models^{\mathfrak{p}} X \neq Y \text{ iff } v(X) \neq v(Y),$$

$$\mathfrak{M}, v \models^{\mathfrak{p}} (\lambda X \psi)\iota Y \varphi \text{ iff there is an } O \in G \text{ such that } \mathfrak{M}, v^X_O \models \psi,$$
$$\mathfrak{M}, v^X_O \models^{\mathfrak{p}} \varphi^Y_X, \text{ and for any } Y\text{-variant } v' \text{ of } v^X_O,$$
$$\text{if } \mathfrak{M}, v' \models^{\mathfrak{p}} \varphi, \text{ then } v'(Y) = O.$$

$$\mathfrak{M}, v \models^{\mathfrak{p}} \neg(\lambda X\psi)\iota Y\varphi \text{ iff for all } O \in G \text{ it holds that } \mathfrak{M}, v^X_O \models^{\mathfrak{p}} \neg\psi, \text{or}$$
$$\mathfrak{M}, v^X_O \models^{\mathfrak{p}} \neg\varphi^Y_X, \text{ or there is an } Y\text{-variant } v' \text{ of } v^X_O,$$
$$\text{such that } \mathfrak{M}, v' \not\models^{\mathfrak{p}} \neg\varphi \text{ and } v'(Y) \neq O$$
$$\mathfrak{M}, v \models^{\mathfrak{p}} \forall X\varphi \text{ iff } \mathfrak{M}, v^X_O \models^{\mathfrak{p}} \varphi, \text{ for all } O \in G,$$
$$\mathfrak{M}, v \models^{\mathfrak{p}} \neg\forall X\varphi \text{ iff } \mathfrak{M}, v^X_O \models^{\mathfrak{p}} \neg\varphi, \text{ for some } O \in G,$$
$$\mathfrak{M}, v \models^{\mathfrak{p}} \exists X\varphi \text{ iff } \mathfrak{M}, v^X_O \models^{\mathfrak{p}} \varphi, \text{ for some } O \in G,$$
$$\mathfrak{M}, v \models^{\mathfrak{p}} \neg\exists X\varphi \text{ iff } \mathfrak{M}, v^X_O \models^{\mathfrak{p}} \neg\varphi, \text{ for all } O \in G.$$

- The notions of a satisfiable formula and a valid formula are defined in a standard way.
- The consequence relation is defined as follows, for all $\Gamma \subseteq \mathscr{F}^2$ and $A \in \mathscr{F}^2$:
  $\Gamma \models_{\mathbf{PRL^2}} \varphi$ iff in every general model $\mathfrak{M}$ and every assignment $v$, if $\mathfrak{M}, v \models \psi$, for all $\psi \in \Gamma$, then $\mathfrak{M}, v \models \varphi$.
- In the completeness proof, we extend the language by individual constants $k, k_1, k_2, \ldots$, and by *relational constants* $K, K_1, K_2, \ldots$.
- These relational constants are introduced to represent fixed $n$-ary relations over individual constants.
- Formally, each relational constant $K^n$ corresponds to a set of $n$-tuples of individual constants, and is interpreted as an element of the general model domain $G \subseteq \mathcal{P}(D^n)$.
- By this concept, we understand the elements of $G$ to be syntactically named via relational constants, allowing us to treat them as concrete surrogates for second-order values during the construction of canonical models.

# Sequent calculus for $\mathbf{RL^2}$

$$(=^2\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, X^{x_1,\dots,x_n}_{b_1,\dots,b_n}, Y^{x_1,\dots,x_n}_{b_1,\dots,b_n} \quad X^{x_1,\dots,x_n}_{b_1,\dots,b_n}, Y^{x_1,\dots,x_n}_{b_1,\dots,b_n}, \Gamma \Rightarrow \Delta}{X = Y, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow=^2)\ \frac{X^{x_1,\dots,x_n}_{a_1,\dots,a_n}, \Gamma \Rightarrow \Delta, Y^{x_1,\dots,x_n}_{a_1,\dots,a_n} \quad Y^{x_1,\dots,x_n}_{a_1,\dots,a_n}, \Gamma \Rightarrow \Delta, X^{x_1,\dots,x_n}_{a_1,\dots,a_n}}{\Gamma \Rightarrow \Delta, X = Y}$$

$$(\exists^2\Rightarrow)\ \frac{\varphi^X_A, \Gamma \Rightarrow \Delta}{\exists X\varphi, \Gamma \Rightarrow \Delta} \quad (\Rightarrow\exists^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^X_B}{\Gamma \Rightarrow \Delta, \exists X\varphi} \quad (\iota^2_1\Rightarrow)\ \frac{\varphi^Y_A, \psi^X_A, \Gamma \Rightarrow \Delta}{(\lambda X\psi)\iota Y\varphi, \Gamma \Rightarrow \Delta}$$

$$(\forall^2\Rightarrow)\ \frac{\varphi^X_B, \Gamma \Rightarrow \Delta}{\forall X\varphi, \Gamma \Rightarrow \Delta} \quad (\iota^2_2\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi^Y_B \quad \Gamma \Rightarrow \Delta, \varphi^Y_C \quad B = C, \Gamma \Rightarrow \Delta}{(\lambda X\psi)\iota Y\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\forall^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^X_A}{\Gamma \Rightarrow \Delta, \forall X\varphi} \quad (\Rightarrow\iota^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^Y_B \quad \Gamma \Rightarrow \Delta, \psi^X_B \quad \varphi^Y_A, \Gamma \Rightarrow \Delta, A = B}{\Gamma \Rightarrow \Delta, (\lambda X\psi)\iota Y\varphi}$$

where $a_1, \dots, a_n$ are fresh individual parameters, not present in $\Gamma$, $\Delta$; $b_1, \dots, b_n$ are arbitrary ind. parameters; $A$ is a fresh relational parameter, not present in $\Gamma, \Delta$, $\varphi$; $B$ and $C$ are arb. rel. parameters.

# Sequent calculus for $\mathbf{PRL^2}$ (part 1)

$$(\Rightarrow\neq^2_1)\ \frac{\Gamma \Rightarrow \Delta, X^{x_1,\ldots,x_n}_{b_1,\ldots,b_n} \quad \Gamma \Rightarrow \Delta, \neg Y^{x_1,\ldots,x_n}_{b_1,\ldots,b_n}}{\Gamma \Rightarrow \Delta, X \neq Y}$$

$$(\Rightarrow\neq^2_2)\ \frac{\Gamma \Rightarrow \Delta, \neg X^{x_1,\ldots,x_n}_{b_1,\ldots,b_n} \quad \Gamma \Rightarrow \Delta, Y^{x_1,\ldots,x_n}_{b_1,\ldots,b_n}}{\Gamma \Rightarrow \Delta, X \neq Y}$$

$$(\neq^2\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg X^{x_1,\ldots,x_n}_{a_1,\ldots,a_n}, Y^{x_1,\ldots,x_n}_{a_1,\ldots,a_n} \quad X^{x_1,\ldots,x_n}_{a_1,\ldots,a_n}, \neg Y^{x_1,\ldots,x_n}_{a_1,\ldots,a_n}, \Gamma \Rightarrow \Delta}{X \neq Y, \Gamma \Rightarrow \Delta}$$

where $a_1, \ldots, a_n$ are fresh individual parameters, not present in $\Gamma$, $\Delta$; $b_1, \ldots, b_n$ are arbitrary ind. parameters; $A$ is a fresh relational parameter, not present in $\Gamma, \Delta$, $\varphi$; $B$ and $C$ are arb. rel. parameters.

# Sequent calculus for **PRL²** (part 2)

$$(\Rightarrow \neg\forall^2)\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi^X_B}{\Gamma \Rightarrow \Delta, \neg\forall X\varphi} \quad (\neg\forall^2 \Rightarrow)\ \frac{\neg\varphi^X_A, \Gamma \Rightarrow \Delta}{\neg\forall X\varphi, \Gamma \Rightarrow \Delta}$$

$$(\neg\exists^2 \Rightarrow)\ \frac{\neg\varphi^X_B, \Gamma \Rightarrow \Delta}{\neg\exists X\varphi, \Gamma \Rightarrow \Delta} \quad (\Rightarrow \neg\exists^2)\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi^X_A}{\Gamma \Rightarrow \Delta, \neg\exists X\varphi}$$

$$(\Rightarrow \neg\iota^2_1)\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi^Y_A, \neg\psi^X_A}{\Gamma \Rightarrow \Delta, \neg(\lambda X\psi)\iota Y\varphi}$$

$$(\Rightarrow \neg\iota^2_2)\ \frac{\neg\varphi^Y_B, \Gamma \Rightarrow \Delta \quad \neg\varphi^Y_C, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, B \neq C}{\Gamma \Rightarrow \Delta, \neg(\lambda X\psi)\iota Y\varphi}$$

$$(\neg\iota^2 \Rightarrow)\ \frac{\neg\varphi^Y_B, \Gamma \Rightarrow \Delta \quad \neg\psi^X_B, \Gamma \Rightarrow \Delta \quad \varphi^Y_A, A \neq B, \Gamma \Rightarrow \Delta}{\neg(\lambda X\psi)\iota Y\varphi, \Gamma \Rightarrow \Delta}$$

where $A$ is a fresh relational parameter, not present in $\Gamma, \Delta$, $\varphi$; $B$ and $C$ are arb. rel. parameters.

**Theorem**

Sequent calculus for $\mathbf{RL^2}$ is sound, complete, and cut-free.

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Sequent calculus for $\mathbf{PRL^2}$ is sound, complete, and cut-free.

Thank you for attention!

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