Propositional Definite Descriptions in Non-Fregean Logic

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Abstract. Definite Descriptions in Non-Fregean Propositional Logics: A Russellian Approach

- In analogy with propositional quantification, we introduce a propositional counterpart of Russell's theory of definite descriptions within the framework of Suszko's non-Fregean logic.
- Unlike first-order definite descriptions, which yield terms denoting objects, our system employs a propositional iota-operator that returns formulas denoting propositions.
- Suszko's hyperintensional logics, inspired by Wittgenstein's view that sentences denote situations rather than truth-values, provide a natural setting for our framework.
- Suszko's connective for propositional identity allows one to distinguish between logically equivalent but non-identical propositions enabling a refined treatment of definite descriptions at the propositional level.
- We present both a semantic and a syntactic account of our theory: through a Kripke-style semantics and a cut-free sequent calculus.

On denoting propositions

In his seminal paper 'On Denoting', Russell introduced the theory of definite descriptions. The theory is general and remains neutral with respect to the kinds of objects to which a definite description may refer, though there are certain restrictions to the theory, such as the following:

> *This is the principle of the theory of denoting I wish to advocate: that denoting phrases never have any meaning in themselves, but that every proposition in whose verbal expression they occur has a meaning. [p. 480]*

Russell, B.: On denoting. Mind **14**, 479–493 (1905)

We study a specific class of definite descriptions, characterized by the fact that they denote propositions. The preceding quotation provides the first example: *the principle of the theory of denoting I wish to advocate*. Looking further back into history, Aristotle, in his *Metaphysics* (Part III of Book IV), uses propositional definite descriptions to denote *the law of non-contradiction*:

> *(...) the most certain principle of all is that regarding which it is impossible to be mistaken; (...) which principle this is, let us proceed to say. It is, that the same attribute cannot at the same time belong and not belong to the same subject and in the same respect; (...) This, then, is the most certain of all principles, since it answers to the definition given above.*

It is noteworthy that probably two of the best-known problems in the theories of knowledge and truth involve definite descriptions referring to specific propositions. Consider, for instance, the Liar Paradox:

The sentence labelled LIAR on this page is false. (LIAR)

where *The sentence labelled LIAR* is a definite description referring to the sentence named LIAR.

Another example is not self-referential and forms a basis for Cartesian epistemology:

The proposition *cogito* is the only proposition of which one can be certain. (COG)

Here we refer to the proposition stating that it cannot be doubted.

Both of these examples are proper definite descriptions, but it is quite easy to think of improper ones:

The only theorem ever proved by Tarski is interesting. (TAR)

- The preceding examples illustrate that denoting phrases signifying propositions are common in ordinary usage.
- Accordingly, they may be treated within the standard logical framework of first-order logic with identity.
- Since logic is topic-neutral, it can be employed to reason about the properties of arbitrary objects—including propositions, which may themselves constitute the *universe of discourse* over which the quantified variables range.
- However, a more natural framework for addressing this class of descriptions seems to be second-order propositional logic, that is, propositional logic enriched with quantifiers binding propositional variables.

- This motivates the proposal of a more specific framework for handling Russellian definite descriptions.
- Another motivation concerns uniqueness, which can be determined in various ways (consider model theory, where expressions such as *unique up to isomorphism* are frequently employed).
- In first-order logic with identity, uniqueness corresponds to co-denotation: two terms denote the same object in the domain of discourse.
- In second-order propositional logic, the biconditional serves as a natural candidate for the uniqueness condition: an object is uniquely determined by a description if every, possibly distinct, object is logically equivalent to the original. However, stricter candidates for connectives determining uniqueness are studied in the context of non-Fregean logics.
- For these reasons, we consider a framework that is both second-order and incorporates a new operation: an identity connective.

Kripke semantics

The language $\mathcal{L}_{\equiv}$ of the systems we are interested in is defined by the following BNF, where p_i is a member of the set of propositional variables V, and $\otimes \in \{\wedge, \vee, \supset\}$:

$$\varphi \ ::= \ p_i \mid \bot \mid \varphi \equiv \varphi \mid \varphi \otimes \varphi \mid \forall p\varphi \mid \exists p\varphi$$

We begin by clarifying the semantics of intuitionistic and classical second-order propositional logics with identity, denoted by ISCI^2_ι and SCI^2_ι, respectively.

- The connective of propositional identity, $\equiv$, was introduced by Suszko in the context of non-Fregean logic.
- According to the early Wittgensteinian view, which Suszko strongly endorsed, sentences denote situations rather than truth values (as in Fregean logic—hence the term non-Fregean).
- Propositional identity thus expresses that two sentences denote the same situation, making it strictly stronger than both intuitionistic and classical equivalence.

Suszko, R.: Abolition of the Fregean axiom. In: Logic Colloquium, pp. 169–239. Springer (1975)

Suszko proposed the following axioms for his basic non-Fregean system, SCI:

($\equiv_1$) $\varphi \equiv \varphi$

($\equiv_2$) $(\varphi \equiv \psi) \supset (\varphi \supset \psi)$

($\equiv_3$) $((\varphi \equiv \psi) \wedge (\chi \equiv \omega)) \supset ((\varphi \otimes \chi) \equiv (\psi \otimes \omega))$

An intuitionistic interpretation of the new connective, grounded in the BHK-interpretation, is proposed in [1]. According to the authors, identity resembles intuitionistic implication in that it denotes a function; however, in the case of identity, this function is not arbitrary but specifically the identity function. The associated conditions are standard for intuitionistic connectives, and a new proof explanation is provided for $\equiv$.

Chlebowski, S., Gałek, M., Tomczyk, A.: Natural deduction systems for intuitionistic logic with identity. Studia Logica **110**, 1381–1415 (2022)

there is no proof of $\bot$	
a is a proof of $\phi \wedge \psi$	$a = (a_1, a_2)$; a_1 is a proof of ϕ and a_2 is a proof of ψ
a is a proof of $\phi \vee \psi$	$a = (a_1, a_2)$; $a_1 = \mathsf{left}$ and a_2 is a proof of ϕ or $a_1 = \mathsf{right}$ and a_2 is a proof of ψ, where left (right) denotes the left (right) disjunct
a is a proof of $\phi \supset \psi$	a is a construction that converts each proof a_1 of ϕ into a proof $a(a_1)$ of ψ
a is a proof of $\phi \equiv \psi$	a is a construction which shows that the classes of proofs of ϕ and ψ are the same or: a is the identity function id

Since the notion of identity is crucial for determining uniqueness in the Russellian approach to definite descriptions, it is advantageous to work within a framework in which two notions of identity of different strengths can be distinguished, namely, equivalence and propositional identity. In addition to identity, we also require propositional quantifiers, whose constructive meaning is given, for example, by Sorensen and Urzyczyn (2006).

a is a proof of $\forall p \varphi$	a is a function transforming every construction of any proposition $\mathbf{p}$ into a proof of $\varphi(\mathbf{p})$
a is a proof of $\exists p \varphi$	a is a pair consisting of a proposition $\mathbf{p}$ (together with its construction) and a proof of $\varphi(\mathbf{p})$

This provides the framework—plausible from both the constructive and the classical points of view—that we shall employ to develop a Russellian theory of propositional definite descriptions.

Sorensen, M. H., Urzyczyn, P.: Lectures on the Curry-Howard Isomorphism. Studies in Logic and the Foundations of Mathematics, vol. 149. Elsevier, Amsterdam (2006)

Second-order non-Fregean frame

By a *second-order non-Fregean frame* we mean an ordered tuple $\mathbf{F} = \langle \mathcal{W}, \mathbb{P}, R, D_{\mathcal{W}} = \{D_w \mid w \in \mathcal{W}\}\rangle$, where $\mathcal{W}$ is a non-empty set, $|\mathbb{P}| \geq 2$, R is a reflexive and transitive binary relation on $\mathcal{W}$ and $D_{\mathcal{W}}$ and each D_w is a family of R-closed subsets of $\mathcal{W}$ satisfying the condition:

$$\text{if } wRw^* \text{ then } D_w \subseteq D_w^*$$

A second-order Kripke frame with reflexive, symmetric and transitive accessibility relation is called *classical second-order Kripke frame*.

Valuation

By a *valuation*, ϑ, in a frame $\mathbf{F} = \langle \mathcal{W}, \mathbb{P}, R, \{D_w \mid w \in \mathcal{W}\} \rangle$ we mean a function that assigns R-closed subsets of $\mathcal{W}$ to propositional variables. A valuation is *admissible* for a world w iff $\vartheta(p) \in D_w$, for all propositional variables p.

Hyper-valuation

By a *hyper-valuation* in a frame $\mathbf{F} = \langle \mathcal{W}, \mathbb{P}, R, \{D_w \mid w \in \mathcal{W}\} \rangle$ we mean a function:

$$\vartheta_h : For \times \mathcal{W} \longrightarrow \mathbb{P}.$$

such that the following conditions are satisfied:

(con) for an arbitrary world w, if $\vartheta_h(\varphi, w) = \vartheta_h(\psi, w)$ and $\vartheta_h(\chi, w) = \vartheta_h(\omega, w)$ then $\vartheta_h(\varphi \otimes \chi, w) = \vartheta_h(\psi \otimes \omega, w)$.

(mon) for arbitrary worlds w, w^* and formulas φ, ψ, if $\vartheta_h(\varphi, w) = \vartheta_h(\psi, w)$ and wRw^*, then $\vartheta_h(\varphi, w^*) = \vartheta_h(\psi, w^*)$.

One may regard the value of φ under ϑ as its *intension*, while the value of φ under ϑ_h represents its *hyperintension*. In the classical context, intension coincides with extension and thus reduces to a truth value, whereas hyperintension corresponds to a situation. In the constructive setting, intension may be understood as the set of information states that force a given formula, while hyperintension is identified with the proof (or collection of proofs) denoted by the formula.

Forcing

Let $\mathbf{F} = \langle \mathcal{W}, \mathbb{P}, R, \{D_w \mid w \in \mathcal{W}\} \rangle$ be a second-order Kripke frame, ϑ be a valuation and ϑ_h a hyper-valuation. A *forcing relation* $\Vdash$ *determined by* ϑ *and* ϑ_h *in* $\mathbf{F}$ is a relation which satisfies, for arbitrary $w \in \mathcal{W}$, the following conditions:

(1) $w, \vartheta, \vartheta_h \Vdash p$ iff $w \in \vartheta(p)$;
(2) $w, \vartheta, \vartheta_h \nVdash \bot$;
(3) $w, \vartheta, \vartheta_h \Vdash \varphi \equiv \psi$ iff $\vartheta_h(\varphi, w) = \vartheta_h(\psi, w)$;
(4) if $w, \vartheta, \vartheta_h \Vdash \varphi \equiv \psi$ then $w, \vartheta, \vartheta_h \Vdash \varphi \supset \psi$;
(5) $w, \vartheta, \vartheta_h \Vdash \varphi \wedge \psi$ iff $w, \vartheta, \vartheta_h \Vdash \varphi$ and $w, \vartheta, \vartheta_h \Vdash \psi$;
(6) $w, \vartheta, \vartheta_h \Vdash \varphi \vee \psi$ iff $w, \vartheta, \vartheta_h \Vdash \varphi$ or $w, \vartheta, \vartheta_h \Vdash \psi$;
(7) $w, \vartheta, \vartheta_h \Vdash \varphi \supset \psi$ iff for all $w' \in R(w)$ we have $w', \vartheta, \vartheta_h \Vdash \varphi$ implies $w', \vartheta, \vartheta_h \Vdash \psi$;
(8) $w, \vartheta, \vartheta_h \Vdash \exists p \varphi$ iff $w, \vartheta(p \mapsto u), \vartheta_h \Vdash \varphi$, for some $u \in D_w$;
(9) $w, \vartheta, \vartheta_h \Vdash \forall p \varphi$ iff for all $w' \in R(w)$ we have $w', \vartheta(p \mapsto u), \vartheta_h \Vdash \varphi$, for all $u \in D_{w'}$.

where for any ϑ, $\vartheta(p \mapsto u)$ is an p-variant of ϑ with u assigned to p.

Model

An ISCI^2_ι-model is a pair $\mathfrak{M} = \langle \mathbf{F}, \Vdash \rangle$, where $\mathbf{F}$ is a second-order non-Fregean frame and $\Vdash$ is a forcing relation. An SCI^2_ι-model is a pair $\mathfrak{M} = \langle \mathbf{F}, \Vdash \rangle$, where $\mathbf{F}$ is a classical second-order Kripke frame.

Complete model

An ISCI^2_ι-model (resp. SCI^2_ι-model) is complete iff for every formula φ, a world w, and for ϑ, if ϑ is admissible for w, then $\{w^* \mid w^*, \vartheta \Vdash \varphi\}$ is an element of D_w. Complete models are natural in the sense that the fact that p denotes a proposition implies that $\neg p$ also denotes some proposition.

Entailment

The notions of a satisfiable formula and a valid formula are defined in a standard way. The consequence relation is defined as follows, for all finite (multi)sets of formulas Γ and Δ, $\Gamma \models_{\mathsf{L}} \Delta$, where $\mathsf{L} \in \{\mathsf{SCI}^2_\iota, \mathsf{ISCI}^2_\iota\}$, iff for every complete L-model, every world w, every valuation ϑ, every hyper-valuation ϑ_h, it holds that if $w, \vartheta, \vartheta_h \Vdash \psi$, for all $\psi \in \Gamma$, then $w, \vartheta, \vartheta_h \Vdash \chi$, for some $\chi \in \Delta$.

Definability of ι operators

The aim of Russell was to give

> *(...) a reduction of all propositions in which denoting phrases occur to forms in which no such phrases occur. [p. 482]*

Retaining Russell's reductionist assumption, the machinery developed so far allows us to define new types of propositional operators, such as: *the only proposition p such that* φ ($\iota p\varphi$). These operators serve merely as abbreviations for longer expressions in the language $\mathcal{L}_{\equiv}$.

- Since our framework is based on non-Fregean principles, it makes sense to consider two types of ι-operators.
- The first one is the standard one — used to refer to a proposition.
- One can call it intensional, since only a standard valuation is needed to define it.
- The other one is hyperintensional, in the sense that it refers not only to the proposition making a certain formula true, but to *the way this proposition is given* (since it refers to the second valuation, the hyper-valuation).
- Let us call this type of description operator *hyperintensional*.

The standard ι-operator can be defined in the following way:

$$\iota p\varphi =_{df} \exists p\big(\varphi \wedge \forall q\big(\varphi(q/p) \supset ((p \supset q) \wedge (q \supset p))\big)\big)$$

One can formulate a direct semantic clause for ι by stipulating

(ι) $w, \vartheta, \vartheta_h \Vdash \iota p\varphi$ iff for all $w' \in R(w)$ there is a unique $u \in D_{w'}$ such that $w', \vartheta(p \mapsto u), \vartheta_h \Vdash \varphi$

In the definition of the standard operator, intuitionistic (or classical) equivalence is used to determine uniqueness.

Hyperintensional ι-operator is defined as follows:

$$\iota^{\equiv} p\varphi =_{df} \exists p\big(\varphi \wedge \forall q(\varphi(q/p) \supset (p \equiv q))\big)$$

The hyperintensional operator is stricter — propositional identity determines uniqueness. A relevant semantic clause is as follows:

($\iota^{\equiv}$) $w, \vartheta, \vartheta_h \Vdash \iota^{\equiv} p\varphi$ iff for all $w' \in R(w)$ there is a unique $\mathsf{b} \in \mathbb{P}$ such that $w', \vartheta, \vartheta_h(p \mapsto \mathsf{b}) \Vdash \varphi$

where for any ϑ_h, $\vartheta_h(p \mapsto \mathsf{b})$ is a p-variant of ϑ_h with the value b assigned to p.

Analogously to the hyperintensional ι-operator, one could also define a second version of quantifiers that modify hyper-valuations:

- $w, \vartheta, \vartheta_h \Vdash \exists^{\equiv} p\varphi$ iff $w, \vartheta, \vartheta_h(p \mapsto \mathsf{b}) \Vdash \varphi$, for some $\mathsf{b} \in \mathbb{P}$;
- $w, \vartheta, \vartheta_h \Vdash \forall^{\equiv} p\varphi$ iff for all $w' \in R(w)$ we have $w', \vartheta, \vartheta_h(p \mapsto \mathsf{b}) \Vdash \varphi$, for all $\mathsf{b} \in \mathbb{P}$.

A detailed study of such quantifiers, however, lies beyond the scope of the present work.

Example 1

Let us fix an arbitrary ISCI^2_ι model and a world w, and consider the formula

$$\iota p(p \supset \bot)$$

which would be forced at w provided there exists a unique (up to equivalence) proposition that makes $p \supset \bot$ true. This is indeed the case: any formula satisfying this condition must be equivalent to a contradiction, both in intuitionistic and in classical logic. Thus the formula is forced at w.
On the other hand consider the formula

$$\iota p(p \supset q)$$

It cannot be true, since there exists no unique proposition $\mathbf{p}$ such that $(p \supset q)$ holds, where $\mathbf{p}$ is assigned to p; both a proposition equivalent to q and a contradiction satisfy this condition.

Example 2

Let us consider an ISCI^2_ι model based on a frame whose proof-assignment at a given world w is injective, i.e., each formula has its unique value under ϑ_h at w. Consider then a formula:

$$\iota^{\equiv}p(p \equiv q)$$

Now, since ϑ_h is injective, there is exactly one formula, such that the value of that formula under ϑ_h, when assigned to p, makes $p \equiv q$ true, and that is the formula q. Thus the considered formula is forced at w. In the same setting consider the formula:

$$\iota^{\equiv}p((p \equiv q) \wedge (p \equiv r))$$

and assume $q \neq r$ (q and r are syntactically distinct). Assuming injective h-assignment, one cannot find a unique formula φ such that $(p \equiv q) \wedge (p \equiv r)$ is true, when $\vartheta_h(\varphi)$ is the value of p under ϑ_h. Thus the formula $\iota^{\equiv}p((p \equiv q) \wedge (p \equiv r))$, being improper description, is not forced at w.

Example 3

Consider now an ISCI^2_ι model in which, at least for a given world w the hyper-valuation is not injective, i.e., that there exist formulas p and q such that $\vartheta_h(p, w) = \vartheta_h(q, w)$. Now we wonder whether formula

$$\iota^{\equiv} r(r \equiv q)$$

is forced at w. Due to the assumption about the hyper-valuation, there is no unique formula φ which makes the formula $(r \equiv q)$ true at w, when the value of φ is the current value of r — at least both p and q fulfill this condition. Therefore the considered formula is false at w.

Extensionality and (hyper-)intensionality

- Suszko maintained that non-Fregean logics are both extensional and two-valued. While they are certainly two-valued (at least those based on classical logic), their extensionality is a more nuanced matter.
- It appears that Suszko wanted to have several irons in the fire: a logic that has the expressive power of some modal systems that are, at the same time, extensional.
- Yet in the language $\mathcal{L}_{\equiv}$, one can construct formulas in which subformulas cannot be replaced by equivalents, that is, formulas sharing the same truth value (as in classical variant) or sharing intension (the set of worlds making a formula true) in intuitionistic logic.
- For example, in (I)SCI the formulas $\neg p$ and $\neg\neg\neg p$ are equivalent in the sense of the underlying biconditional, yet they cannot be interchanged in arbitrary contexts.

Consider

$$\forall p(\neg p \equiv \neg p)$$

which is valid in both logics. But the effect of replacement

$$\forall p(\neg p \equiv \neg\neg\neg p)$$

is not valid, since there are models with world w such that $\vartheta_h(\neg p, w) \neq \vartheta_h(\neg\neg\neg p, w)$. Thus, sharing the same set of truth-making information states is insufficient for replacement. One can say then that these logics are not extensional, and even intensional, with respect to the underlying biconditional. However, they are extensional if equality of hyperintensions is presupposed in replacement (which is the sense in which Suszko employed the term): Given some model $\mathfrak{M}$, if $\mathfrak{M} \Vdash \chi(\varphi)$ and $\mathfrak{M} \Vdash \varphi \equiv \psi$ then $\mathfrak{M} \Vdash \chi(\psi)$. In contemporary terminology, the logics in question would more aptly be described as hyperintensional.

Proof theory. Soundness, completeness, and cut admissibility

- A sequent is an ordered pair written as $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas.
- We write $\models_{\mathsf{L}} \Gamma \Rightarrow \Delta$, where $\mathsf{L} \in \{\mathsf{SCI}^2_\iota, \mathsf{ISCI}^2_\iota\}$, iff $\Gamma \models_{\mathsf{L}} \Delta$.
- If $\mathcal{S}$ is a set of sequents and S is a sequent, we write $\mathcal{S} \models_{\mathsf{L}} S$ iff $\models_{\mathsf{L}} T$, for all $T \in \mathcal{S}$, implies $\models_{\mathsf{L}} S$.
- If $\mathcal{S} \vdash_{\mathsf{L}} S$ and each cut is on a formula that belongs to $\mathcal{S}$, then we write $\mathcal{S} \vdash^{cf}_{\mathsf{L}} S$.

Sequent calculus for SCI^2_ι (part 1)

$$(\mathrm{Ax}_\varphi)\ \ \varphi, \Gamma \Rightarrow \Delta, \varphi \qquad (\mathrm{Ax}_\bot)\ \ \bot, \Gamma \Rightarrow \Delta$$

$$(\mathrm{Cut})\ \ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\mathrm{C}{\Rightarrow})\ \ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\mathrm{C})\ \ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\equiv\Rightarrow_1)\ \ \frac{\varphi \equiv \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (\equiv\Rightarrow_2)\ \ \frac{\varphi \supset \psi, \Gamma \Rightarrow \Delta}{\varphi \equiv \psi, \Gamma \Rightarrow \Delta}$$

$$(\equiv\Rightarrow_3)\ \ \frac{(\varphi \otimes \chi) \equiv (\psi \otimes \omega), \Gamma \Rightarrow \Delta}{\varphi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta}$$

where $\otimes \in \{\wedge, \vee, \supset\}$

Sequent calculus for SCI^2_ι (part 2)

$$(\vee \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\wedge \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\supset\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \supset \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\Rightarrow\supset)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$(\forall \Rightarrow)\ \frac{\varphi^p_\psi, \Gamma \Rightarrow \Delta}{\forall p \varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \forall)\ \frac{\Gamma \Rightarrow \Delta, \varphi^p_s}{\Gamma \Rightarrow \Delta, \forall p \varphi}$$

$$(\exists \Rightarrow)\ \frac{\varphi^p_s, \Gamma \Rightarrow \Delta}{\exists p \varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \exists)\ \frac{\Gamma \Rightarrow \Delta, \varphi^p_\psi}{\Gamma \Rightarrow \Delta, \exists p \varphi}$$

in $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$, s is not present in φ, Γ, and Δ.

Sequent calculus for SCI^2_ι (part 3)

$$(\iota^{\equiv}\Rightarrow_1)\quad \frac{\varphi^p_s, \Gamma \Rightarrow \Delta}{\iota^{\equiv} p\varphi, \Gamma \Rightarrow \Delta}$$

$$(\iota^{\equiv}\Rightarrow_2)\quad \frac{\Gamma \Rightarrow \Delta, \varphi^p_\psi \quad \Pi \Rightarrow \Sigma, \varphi^p_\chi \quad \psi \equiv \chi, \Theta \Rightarrow \Lambda}{\iota^{\equiv} p\varphi, \Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}$$

$$(\Rightarrow \iota^{\equiv})\ \frac{\Gamma \Rightarrow \Delta, \varphi^p_\psi \quad \varphi^p_\chi, \Pi \Rightarrow \Sigma, \psi \equiv \chi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \iota^{\equiv} p\varphi} \qquad (\iota \Rightarrow_1)\ \frac{\varphi^p_s, \Gamma \Rightarrow \Delta}{\iota p\varphi, \Gamma \Rightarrow \Delta}$$

$$(\iota \Rightarrow_2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^p_\psi \quad \Pi \Rightarrow \Sigma, \varphi^p_\chi \quad \Theta \Rightarrow \Lambda, \psi, \chi \quad \psi, \chi, \Upsilon \Rightarrow \Phi}{\iota p\varphi, \Gamma, \Pi, \Theta, \Upsilon \Rightarrow \Delta, \Sigma, \Lambda, \Phi}$$

$$(\Rightarrow \iota)\ \frac{\Gamma \Rightarrow \Delta, \varphi^p_\psi \quad \varphi^p_\chi, \psi, \Pi \Rightarrow \Sigma, \chi \quad \varphi^p_\chi, \chi, \Theta \Rightarrow \Lambda, \psi}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda, \iota p\varphi}$$

in $(\iota^{\equiv}\Rightarrow_1)$, and $(\iota \Rightarrow_1)$, s is not present in Γ, Δ, and φ; in $(\Rightarrow \iota^{\equiv})$, and $(\Rightarrow \iota)$, χ is not present in Γ, Π, Σ, Δ, Θ, Λ, φ, and ψ.

The sequent calculus for ISCI_ι^2 is obtained from the one for SCI_ι^2 by the following restriction of sequents: in a sequent $\Gamma \Rightarrow \Delta$, we write Δ for an empty set or a single formula. Additionally, the rule $(\Rightarrow$C) is removed and the rules $(\Rightarrow \vee)$ and $(\iota \Rightarrow_2)$ are replaced with the following ones, where χ is not present in $\Gamma, \Pi, \Theta, \Upsilon$, φ, and ψ:

$$(\Rightarrow \vee_1)\ \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad (\iota \Rightarrow_2^1)\ \frac{\Gamma \Rightarrow \varphi^p_\psi \quad \Pi \Rightarrow \varphi^p_\chi \quad \Theta \Rightarrow \psi \quad \psi, \chi, \Upsilon \Rightarrow \omega}{\iota p \varphi, \Gamma, \Pi, \Theta, \Upsilon \Rightarrow \omega}$$

$$(\Rightarrow \vee_2)\ \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad (\iota \Rightarrow_2^2)\ \frac{\Gamma \Rightarrow \varphi^p_\psi \quad \Pi \Rightarrow \varphi^p_\chi \quad \Theta \Rightarrow \chi \quad \psi, \chi, \Upsilon \Rightarrow \omega}{\iota p \varphi, \Gamma, \Pi, \Theta, \Upsilon \Rightarrow \omega}$$

We adopt sequent rules for propositional quantifiers in the style of [2], and we formulate the rules for $\equiv$ in the spirit of [1].

Chlebowski, S., Leszczyńska-Jasion, D.: An investigation into intuitionistic logic with identity. Bulletin of the Section of Logic **48**(4), 259–283 (2019)

Sorensen, M. H., Urzyczyn, P.: Lectures on the Curry-Howard Isomorphism. Studies in Logic and the Foundations of Mathematics, vol. 149. Elsevier, Amsterdam (2006)

The following sequents are provable in the sequent calculi for ISCI_ι^2 and SCI_ι^2:

- $\iota p\varphi \Rightarrow \exists p\big(\varphi \wedge \forall q\big(\varphi^p_q \supset ((p \supset q) \wedge (q \supset p))\big)\big)$
- $\exists p\big(\varphi \wedge \forall q\big(\varphi^p_q \supset ((p \supset q) \wedge (q \supset p))\big)\big) \Rightarrow \iota p\varphi$
- $\iota^{\equiv} p\varphi \Rightarrow \exists p\big(\varphi \wedge \forall q\big(\varphi^p_q \supset (p \equiv q)\big)\big)$
- $\exists p\big(\varphi \wedge \forall q\big(\varphi^p_q \supset (p \equiv q)\big)\big) \Rightarrow \iota^{\equiv} p\varphi$

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\iota^{\equiv} p\varphi, \varphi^p_s \Rightarrow \varphi^p_s
\qquad
\dfrac{
\dfrac{
\dfrac{
\varphi^p_r \Rightarrow \varphi^p_r \qquad \varphi^p_s \Rightarrow \varphi^p_s \qquad s \equiv r \Rightarrow s \equiv r
}{
\iota^{\equiv} p\varphi, \varphi^p_s, \varphi^p_r \Rightarrow s \equiv r
}\;(\iota^{\equiv} \Rightarrow_2)
}{
\iota^{\equiv} p\varphi, \varphi^p_s \Rightarrow \varphi^p_r \supset (s \equiv r)
}\;(\Rightarrow\supset)
}{
\iota^{\equiv} p\varphi, \varphi^p_s \Rightarrow \forall q\big(\varphi^p_q \supset (s \equiv q)\big)
}\;(\Rightarrow \forall)
}{
\dfrac{
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\iota^{\equiv} p\varphi, \varphi^p_s \Rightarrow \varphi^p_s \wedge \forall q\big(\varphi^p_q \supset (s \equiv q)\big)
}{
\iota^{\equiv} p\varphi, \varphi^p_s \Rightarrow \exists p\Big(\varphi \wedge \forall q\big(\varphi^p_q \supset (p \equiv q)\big)\Big)
}\;(\Rightarrow \exists)
}{
\iota^{\equiv} p\varphi, \iota^{\equiv} p\varphi \Rightarrow \exists p\Big(\varphi \wedge \forall q\big(\varphi^p_q \supset (p \equiv q)\big)\Big)
}\;(\iota^{\equiv} \Rightarrow_1)
}{
\iota^{\equiv} p\varphi \Rightarrow \exists p\Big(\varphi \wedge \forall q\big(\varphi^p_q \supset (p \equiv q)\big)\Big)
}\;(\mathrm{C}\Rightarrow)
}\;(\Rightarrow \wedge)
$$

$$
\dfrac{\dfrac{\dfrac{\varphi^p_s \Rightarrow \varphi^p_s \qquad \dfrac{\dfrac{\varphi^p_q \Rightarrow \varphi^p_q \qquad s \equiv q \Rightarrow s \equiv q}{\varphi^p_q, \varphi^p_q \supset (s \equiv q) \Rightarrow s \equiv q}\,(\supset\Rightarrow)}{\varphi^p_q, \forall q\big(\varphi^p_q \supset (s \equiv q)\big) \Rightarrow s \equiv q}\,(\forall \Rightarrow)}{\varphi^p_s, \forall q\big(\varphi^p_q \supset (s \equiv q)\big) \Rightarrow \iota^{\equiv} p\varphi}\,(\Rightarrow \iota^{\equiv})}{\varphi^p_s \wedge \forall q\big(\varphi^p_q \supset (s \equiv q)\big) \Rightarrow \iota^{\equiv} p\varphi}\,(\wedge \Rightarrow)}{\exists p\big(\varphi \wedge \forall q\big(\varphi^p_q \supset (p \equiv q)\big)\big) \Rightarrow \iota^{\equiv} p\varphi}\,(\exists \Rightarrow)
$$

Theorem. Strong soundness

Let $\mathsf{L} \in \{\mathsf{SCI}^2_\iota, \mathsf{ISCI}^2_\iota\}$. Let $\mathcal{S}$ be a set of sequents and S be a sequent. If $\mathcal{S} \vdash_{\mathsf{L}} S$, then $\mathcal{S} \models_{\mathsf{L}} S$.

Theorem. Strong completeness

Let $\mathsf{L} \in \{\mathsf{SCI}^2_\iota, \mathsf{ISCI}^2_\iota\}$. Let $\mathcal{S}$ be a set of sequents and S be a sequent. If $\mathcal{S} \models_{\mathsf{L}} S$, then $\mathcal{S} \vdash^{cf}_{\mathsf{L}} S$.

Theorem. Cut admissibility

Let $\mathsf{L} \in \{\mathsf{SCI}^2_\iota, \mathsf{ISCI}^2_\iota\}$. For every sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{\mathsf{L}} \Gamma \Rightarrow \Delta$, then there exists a cut-free derivation of $\Gamma \Rightarrow \Delta$ in a sequent calculus for L.

Concluding remarks

- We have presented a framework for addressing Russellian propositional definite descriptions.
- The framework is applicable in both classical and constructive settings and can be easily adjusted to even weaker systems.
- The incorporation of propositional identity provides an additional means of determining uniqueness.

Further research

- Since the logics considered here are comparatively weak—in the sense that only trivial identities of the form $\varphi \equiv \varphi$ are provable—it would be of interest to investigate stronger systems of identity, such as WB (Boolean non-Fregean logic extending SCI) or, in the intuitionistic context, the logic of propositional isomorphism.
- One could investigate further the quantifiers $\forall^{\equiv}$ and $\exists^{\equiv}$.
- One could develop propositional definite descriptions within approaches alternative to Russell's.
- One could attempt a constructive, purely syntactic proof of the cut-admissibility theorem for the sequent calculi presented here.
- One could also investigate proof-theoretic formalisms other than sequent calculi, such as natural deduction and analytic tableaux.

Thank you for your attention!

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