

A Theory of Definite Descriptions

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Visiting Speaker Seminar
Dept of Philosophy
University of Stirling
09 April 2024

Definite Descriptions. I

A definite description is an expression of the form 'the F ': the first man on the moon, the author of *Principia Mathematica*, the person lolling on my couch.

'The F ' purports to refer to a unique object that answers to the description. If there is exactly one F , the definite description 'the F ' is called *proper*, if not, it is *improper*.

'The first man on the moon' is proper, because there is one and only one person answering to it: Neil Armstrong.

'The author of *Principia Mathematica*' is improper, because this book is a joint work by Whitehead and Russell.

'The person lolling on my couch' is improper (or so I hope), because there is no one lolling on my couch right now.

Grammatically definite descriptions are singular terms. They are complex names for particulars. Sentences of the form 'The F is G ' are formed from a singular term by attaching a predicate.

Definite Descriptions. II

It may not always be evident whether a definite description is proper or improper, and there may be widespread erroneous belief whether a definite description is proper or improper:

- (1) Who is the most famous detective? Is it Sherlock Holmes or some real detective, we know not who?
- (2) Who is the most famous musketeer? Charles de Batz de Castelmore, Comte d'Artagnan, or the character from Dumas?
- (3) Who is the present King of England? Many people think it is Charles III, but there has been no King of England since William III.

This may be quite an interesting case: an attempt to refer to Charles III. with an utterance containing 'the present King of England' will almost certainly succeed. (There is some such convention.) Yet the description is improper.

Logic does not tell us whether a definite description is proper or improper. But if 'the' is a logical expression, logic should say what their inferential relations are. This is what I am interested in.

The Formalisation of Definite Descriptions. I

Definite descriptions are commonly formalised by the ι operator. It binds a variable and forms a singular term from a formula:

(1) The F

(2) $\iota x F$

Sentences containing definite descriptions are then formalised by attaching ι terms to predicate letters:

(3) The F is G

(4) $G \iota x F$

The ι operator was introduced by Peano.

Russell's Theory of Definite Descriptions. I

The most celebrated theory of definite descriptions is Russell's – often just referred to by the definite description 'the Theory of Definite Descriptions'. It was first proffered in 'On Denoting'. The meaning of 'the' in the strict sense that concerns Russell involves uniqueness. This means that the following mean the same:

(1) The father of Charles II. was executed.

(2) It is not always false of x that x begat Charles II. and that x was executed and that 'if y begat Charles II., y is identical with x ' is always true of x .

Or more briefly:

(3) One and only one person begat Charles II. and he was executed.

Note what happens to 'the': it disappears upon analysis.

Russell's Theory of Definite Descriptions. I

According to Russell, it is only apparent that definite descriptions are singular terms. Because of the possibility of forming improper definite descriptions, Russell argued that we must not look for the meanings of definite descriptions in isolation, but only in the context of complete sentences in which they occur. (Compare with Frege's *Context Principle*.) And then they disappear upon analysis:

'The present King of France is bald'

means

'Exactly one person is presently King of France and he is bald.'

Sentences containing definite descriptions are analysed in terms of quantification, identity and propositional connectives.

'The F ' has no meaning in itself, but any sentence in which it occurs is given a meaning by Russell's analysis.

Russell's Theory of Definite Descriptions. II

Russell uses Peano's ι operator, too, but with a caveat. In *Principia Mathematica* its use is defined by a contextual definition in complete sentences that involves scope markers:

$$*14.01 \quad [\iota x F]G(\iota x F) =_{df.} \exists x(\forall y(F_y^x \leftrightarrow x = y) \wedge G)$$

Arthur Prior, followed by Neale and others, observed that for Russell definite descriptions are akin to quantificational phrases.

The question whether 'the F ' is proper or improper is as misguided as the question 'Which particulars are named by 'everything', 'something', 'nothing'?

What can be asked is whether 'The F exists' is true or false.

Russell's Theory of Definite Descriptions. III

Russell defines expressions $\iota x F$ (the F) by a contextual definition. This requires a means for drawing scope distinctions, e.g. to distinguish the internal and the external negations of 'The present King of France is bald'. They differ in truth conditions:

- (1) The present King of France is not bald: false
- (2) It is not the case that the present King of France is bald: true.

Russell represents the difference by repeating the definite description in square brackets to mark its scope:

- (1) $[\iota x Fx] \neg G(\iota x Fx)$ unpacks as $\exists x(\forall y(Fy \leftrightarrow x = y) \wedge \neg Gx)$
- (2) $\neg[\iota x Fx] G(\iota x Fx)$ unpacks as $\neg\exists x(\forall y(Fy \leftrightarrow x = y) \wedge Gx)$

This is admittedly rather clumsy notation.

Definite Descriptions formalised by Binary Quantification

Arthur Prior introduced a more elegant option of formalising complete sentences 'The F is G ' by a binary quantifier as $Ix(Fx, Gx)$. Neale and others write $(\iota x Fx)(Gx)$, but these are notational variants: they are intended to mean the same thing.

This allows for a perspicuous representation of scope distinctions:

- 1 The F is G : $Ix(Fx, Gx)$
- 2 Internal negation: The F is not G : $Ix(Fx, \neg Gx)$
- 3 External negation: It's not the case that the F is G : $\neg Ix(Fx, Gx)$
- 4 Modality *de re*: The F is necessarily G : $Ix(Fx, \Box Gx)$
- 5 Modality *de dicto*: It is necessary that the F is G : $\Box Ix(Fx, Gx)$

Note: it is sometimes questioned whether scope markers are necessary in the case of negation, but I know of no-one who thinks they are not necessary in modal logic: there are instances of 4 and 5 that differ in truth value: 'The present King of France is possibly bald' is false, 'It is possible that the present King of France is bald' may well be true, had France stayed a monarchy.

The Aim of the Rest of this Talk

I shall adopt a largely Russellian approach to definite descriptions. I shall be unRussellian in using positive free, not classical, logic. The reason is that I am going to apply my approach in quantified modal logic, and there is some consensus that here a positive free logic is the best choice for the underlying non-modal logic.

I'll put forward rules of inference for a binary quantifier I that formalises sentences of the form 'The F is G ' by $Ix(F, G)$ that are in accordance with proof-theoretic semantics (see next slide). Russell, incidentally, considered the definite article to be an expression of logic: it is analysed in terms of the logical primitives.

My rules can be added to classical logic and provide a proof theory for an account of definite descriptions as quantificational phrases.

They could be added to negative free logic, but to stay within the spirit of negative free logic, they require modification.

Proof-Theoretic Semantics

Proof-theoretic semantics is founded on a fertile remark of Gentzen's relating to his systems of natural deduction, developed further by Dummett and Prawitz. Schroeder-Heister applies their ideas also to sequent calculus, but I'll focus on natural deduction.

According to proof-theoretic semantics, the meanings of logical constants are defined by their introduction and elimination rules, if they are *in harmony* (more precisely, *stable*), as Dummett says.

A necessary (but not sufficient) condition is that rules in natural deduction permit normalisation: detours through deducing a formula by an introduction rules that is then used as the major premise of an elimination rule can be removed from deductions.

The philosophical motivation for this is that the elimination rules permit the deduction of *no more* consequences than are justified by the grounds as specified by the introduction rules.

Free Logic

Classical logic demands that domains of quantification are non-empty and that singular terms refer. Classical logic makes certain existence assumptions. The following are classically valid:

- (1) For any term t : $\exists x x = t$: ' t exists'.
- (2) $\neg \exists x Fx \rightarrow \exists x \neg F$: if nothing is F , then something is not- F .

The development of free logic by Jaśkowski, Hintikka and Lambert was motivated by avoiding these existence assumptions and by permitting empty domains of quantification and reference failure.

- (1) 'Pegasus exists' should not be logically true.
- (2) If nothing exists, then nothing is an F , whatever F might be, and so it should not be the case that something is not- F .

Quantifiers, however, have existential import: although it may be empty, the domain of quantification contains only what exists.

Positive free logic permits atomic formulas containing non-referring terms to be true, in particular $t = t$ is a logical law.

Quantification in Positive Free Logic

Quantification is relativised to an 'existence predicate':

$$\begin{array}{c} [\exists!a]^i \\ \Pi \\ (\forall I) \quad \frac{A_a^x}{\forall x A} ; \text{ where } a \text{ is fresh} \end{array} \qquad (\forall E) \quad \frac{\forall x A \quad \exists!t}{A_t^x}$$

The standard semantics for free logic employs two domains:

- (1) The 'inner domain': the domain of things that exists, over which the quantifiers range.
- (2) The 'outer domain': the domain of 'things' that do not exist, outside the range of the quantifiers.

Indulging in this Meinongian fantasy is at odds with the initial motivation of free logic, but has become quite common.

Note: $\exists!$ is not necessary: Jaśkowski permits terms by themselves to occur in rules to indicate that they are supposed to denote.

Identity and Definite Descriptions in Positive Free Logic

The theory of identity of positive free logic is standard:

$$\begin{array}{ll} (= I) & \overline{t = t} \\ (= E) & \frac{t_1 = t_2 \quad A_{t_1}^x}{A_{t_2}^x} \end{array}$$

Positive free logic is motivated by the thought that atomic formulas containing non-referring terms can be true. In particular, $t = t$ is true for any t , whether referring or not.

On the present account, definite descriptions are not singular terms. So we cannot replace definite descriptions for t in $t = t$.

However, I take on a thought that motivated the formulation of theories of definite description in free logic by Hintikka and Lambert: 'The F is G ' may be true even if no unique F exists.

Examples: 'The king of England is bald'; 'The tallest of the seven dwarfs is smaller than Snow White' (example suggested to me by Niko Strobach); 'The Cheshire Cat grins' (assuming 'Cheshire Cat' to be a kind of cat, even if there is only one exemplar in the story).

The Meaning of I . I

What are sensible conditions under which 'The F is G ' true, if the core ideas of positive free logic are taken on board?

Proposal: $Ix[F, G]$ is true iff there is a unique F and it is G , but this F need not exist (i.e. the unique F may be in the outer domain, no preference is given to the inner domain).

Thus 'The author of *Principia* smokes a pipe', 'The round square is round' and 'The round square is not round' are all false.

Note: The rules on the next slide could be added to classical logic. Then $Ix[F, G]$ is equivalent to $\exists x(\forall y(F_y^x \leftrightarrow x = y) \wedge G)$, that is, they give a formalisation of Prior's, Neale's and others' account of definite descriptions as quantificational phrases.

Rules for I suitable to negative free logic require the addition of existence assumptions, but this is possible, too.

Natural Deduction Rules for I for Positive Free Logic

$$(II) \quad \frac{F_t^x \quad G_t^x \quad \begin{array}{c} [F_a^x]^i \\ \Pi \\ a = t \end{array}}{Ix(F, G)} i$$

where a is different from t and does not occur in F , G or any undischarged assumptions of Π except F_a^x .

$$(IE^1) \quad \frac{Ix(F, G) \quad \begin{array}{c} \underbrace{[F_a^x]^i, [G_a^x]^j} \\ \Pi \\ C \end{array}}{C} i, j$$

where a is not free in C or any undischarged assumptions in Π except F_a^x , G_a^x .

$$(IE^2) \quad \frac{Ix(F, G) \quad F_{t_1}^x \quad F_{t_2}^x}{t_1 = t_2}$$

The Meaning of I . II

The binary quantifier I is governed by harmonious rules. The grounds for asserting $Ix(F, G)$ as stipulated by (II) justify the consequences that can be drawn from it by (IE_1) and (IE_2) . A normalisation theorem can be proved for positive free logic with I . The rules for I satisfy a necessary condition for proof-theoretic semantics. Assuming they also satisfy other conditions (which I think they do), the meaning of I is defined by its rules.

The meaning of I requires the uniqueness of F , but not its existence: 'The F is G ' is true iff a unique F is G , but the unique F need not exist. Otherwise it is false.

In the standard dual domain semantics: 'The F is G ' is true iff there is a unique F in the (inner or outer) domain, and it is G .

Consequently 'The F is identical to the F ', $Ix(F, Iy(F, x = y))$, is false if ' Fa ' is true for more than one name ' a ' (referring or not).

Definite Descriptions as Singular Terms in Free Logic

The standard account of definite descriptions in free logic is to treat them as singular terms and to formalise them with a term-forming ι operator, which is governed by:

Lambert's Axiom: $\forall y(y = \iota x A \leftrightarrow \forall x(A \leftrightarrow x = y))$

Anything is identical to the A iff it is the unique existing A .

As the quantifiers range only over what exists, Lambert's axiom says nothing at all about improper definite descriptions. Using the axiom requires that $\iota x A$ is identical to something that exists or that something that exists is a unique A .

Lambert adds a further axiom, which gives his preferred theory of definite descriptions: $t = \iota x(x = t)$

t is identical to the thing that is t .

Some Properties of I in Positive Free Logic

The derivability of a formula analogous to $\neg x(x = t) = t$ is immediate. Let F and G be $x = t$ in (II):

$$\frac{t = t \quad t = t \quad [a = t]^1}{Ix[x = t, x = t]} 1$$

This is desirable: It is an important principle for Lambert, but he has to add it as an axiom. Here it is for free.

An analogue of one half of Lambert's Axiom is derivable.

$$\frac{\frac{[a = b]^1 \quad \frac{[c = b]^3 \quad [A_c^x]^4}{A_b^x}}{A_a^x} \quad \frac{Ix[A, x = b] \quad \frac{[A_a^x]^2 \quad \frac{[c = b]^3 \quad [A_c^x]^4}{A_b^x}}{a = b}}{A_a^x \leftrightarrow a = b} 1,2}{\frac{Ix[A, x = b] \quad \frac{A_a^x \leftrightarrow a = b}{\forall x(A \leftrightarrow x = b)}}{\forall x(A \leftrightarrow x = b)} 3,4$$

The other half is not derivable. The unique existence of an A (which is b) is not sufficient for $Ix[A, x = b]$, because there might also be a non-existent A , in which case $Ix[A, x = b]$ is false.

Propositional Modal Logic

The rules for the connectives $\neg, \rightarrow$ are those for classical logic.
The rules for propositional modal logic **K** are standard:

$$\begin{array}{ccc} \begin{array}{c} [A]^i \\ \Pi \\ B \\ (\rightarrow I) \frac{A \rightarrow B}{A \rightarrow B} i \end{array} & \begin{array}{c} (\rightarrow E) \frac{A \rightarrow B \quad A}{B} \end{array} & \begin{array}{c} [\neg A]^i \\ \Pi \\ \perp \\ (\perp E^C) \frac{\perp}{A} i \end{array} \\ \\ & \begin{array}{c} \underbrace{[A_1]^{i_1} \dots [A_n]^{i_n}} \\ \Pi \\ B \\ (\Box I/E) \frac{\Box A_1 \dots \Box A_n \quad B}{\Box B} i_1, \dots, i_n \end{array} & \text{where } n \geq 0 \end{array}$$

T, S4, S5, etc.: add the usual axioms/rules/modifications of rules.

Restrictions on Quantification and Identity

To simplify matters, I'll assume that the only terms are constants and parameters and they are interpreted rigidly, i.e. as designating the same object in every world.

This absolves us from restricting $(\forall E)$ and $(= E)$:

$$(\forall E) \quad \frac{\forall x A \quad \exists! t}{A_t^x} \qquad (= E) \quad \frac{s = t \quad A_s^x}{A_t^x}$$

s and t being rigid ensures these rules are sensible in quantified modal logic. (Consider the case where x is in the scope of a modal operator in A and the instantiating terms are not rigid.)

This means the following rules hold:

$$(RC_1) \quad \frac{s = t}{\Box s = t} \qquad (RC_2) \quad \frac{\neg s = t}{\Box \neg s = t}$$

Natural Deduction Rules for $/$

$$(II) \quad \frac{F_t^x \quad G_t^x \quad \begin{array}{c} [F_a^x]^i \\ \Pi \\ a = t \end{array}}{Ix[F, G]}_i$$

where a is different from t and does not occur in F , G or any undischarged assumptions of Π except F_a^x .

$$(IE^1) \quad \frac{Ix[F, G] \quad \begin{array}{c} \underbrace{[F_a^x]^i, [G_a^x]^j} \\ \Pi \\ C \end{array}}{C}_{i,j}$$

where a is not free in C or any undischarged assumptions in Π except F_a^x , G_a^x .

$$(IE^2) \quad \frac{Ix[F, G] \quad F_{t_1}^x \quad F_{t_2}^x}{t_1 = t_2}$$

Constants and parameters being the only terms, t, t_1, t_2 are rigid.

Reasons for the Restriction to Rigid Terms. Ia. (II)

An example adapted from Pavel Tichý's work:

'The Sovereign' is a non-rigid designator for the monarch of the United Kingdom. 'The Sovereign is monarch of the United Kingdom' is true, and anyone who is monarch of the United Kingdom is identical to the Sovereign. Rules of accession to the British Throne mean that the role of Sovereign is always occupied. So: (1) It is constitutionally necessary that someone is (identical to) the Sovereign. The premises of (II) are then fulfilled. But it is not the case that (2) the monarch of the United Kingdom is such that it is constitutionally necessary that someone is (identical to) him. 'Being identical to something' expresses existence: King Charles III does not exist by constitutional necessity.

Premise (1) is *de dicto*; the conclusion (2) is *de re*.

Reasons for the Restriction to Rigid Terms. Ib. (II)

$$(II) \quad \frac{F_t^x \quad G_t^x \quad \frac{[F_a^x]^i \quad \Pi}{a = t}}{Ix[F, G]}_i$$

where a is different from t and does not occur in F , G or any undischarged assumptions of Π except F_a^x .

t : the Sovereign; Fx : x is monarch of the UK; $\Box$: It is constitutionally necessary that; Gx : $\Box \exists y \ y = x$

The these are true:

Ft : t is monarch of the UK

$\Box \exists y \ y = t$: It is constitutionally necessary that someone is identical to t .

$\forall z(Fz \rightarrow z = t)$: Anyone who is monarch of the UK is identical to t .

But this is not true:

$Ix(Fx, \Box \exists y \ y = x)$: The monarch of the UK is such that it is constitutionally necessary that someone is identical to him.

Reasons for the Restriction to Rigid Terms. IIa. (IE_2)

Another adaptation of an example of Tichý's: Suppose some things in some worlds are ordered from most miserable to most awesome, so that there is always a most miserable thing and a most awesome thing in a world. Call the latter 'Its Awesomeness' and the former 'Its Miserableness'. So (1) 'It is necessary that Its Miserableness exists' and (2) 'It is necessary that Its Awesomeness exists' are true. Clearly, Its Awesomeness need not be identical to Its Miserableness. So let w be a world in which they are not.

Assume also that amongst these things in these worlds exactly one exists in all of them and that in w it is mediocre, i.e. (3) 'The necessarily existing thing is mediocre' is true at w . (In fact it does not matter what property it is assumed to have in w .)

Then the premises of (IE_2) are fulfilled, but its conclusion is not.

In (1) and (2) the modality is *de dicto*, while in (3) it is *de re*.

Reasons for the Restriction to Rigid Terms. IIa. (IE_2)

$$(IE^2) \quad \frac{Ix[F, G] \quad F_{t_1}^x \quad F_{t_2}^x}{t_1 = t_2}$$

Consider a bunch of worlds so that in every world, there is exactly one most awesome thing and exactly one most miserable thing, which are different in the world w we are considering now, and exactly one thing exists in every world, and in w it is mediocre.

t_1 : Its Awesomeness; t_2 : Its Miserableness

Fx : $\Box\exists y \ y = x$, Gx : x is mediocre.

Then these are true:

$Ix[\Box\exists y \ y = x, Gx]$: The necessarily existing thing is mediocre.

$\Box\exists y \ y = t_1$: It is necessary that Its Awesomeness exists.

$\Box\exists y \ y = t_2$: It is necessary that Its Miserableness exists.

But $t_1 = t_2$ is false.

Semantics for /

$\mathcal{M}, w \Vdash_v Ix[A, B]$ iff there is a $d \in \mathcal{D}$ such that for the x -variant v' of v such that $v'(x) = d$, $\mathcal{M}, w \Vdash_{v'} A$, and for any x -variant v'' of v' , if $\mathcal{M}, w \Vdash_{v''} A$, then $v''(x) = d$, and $\mathcal{M}, w \Vdash_{v'} B$.

So: $Ix[F, G]$ is true at a world w iff F is assigned a singleton subset of the domain of the model at w and its element is G at w .

Note: I haven't proved completeness for modal logic, but the conditions are sound and complete in non-modal logic.

Uniqueness of F in the domain of the model is quite a strong requirement. If every $x \in \mathcal{D}$ is in some $\mathcal{D}(w)$, then if M is the universal possibility $Ix[F, G] \models Ix[F, M\exists!x]$.

Alternative: assign to each w an outer domain that is a subset of the domain of the model and require uniqueness only there. Not sure it would make a difference: we haven't got outer quantifiers or a predicate true only of the things in the outer domain ('subsists').

Rigidity. I

A term is rigid in w iff it picks out the same object in every world accessible from w . What does rigidity mean in my system?

Let's restrict A to predicate letters. $Ix[A, B]$ is about the A rigidly if the same object that is the A that is B in w is the A in every accessible world. 'The A is necessarily the A ' expresses the rigidity of 'the A ' in terms of binary quantification: $Ix[A, \Box Iy[A, x = y]]$.

$\mathcal{M}, w \Vdash_v Ix[A, \Box Iy[A, x = y]]$: $I(A, w)$ is a singleton $\{a\}$, and in every accessible world w' , $I(A, w') = \{a\}$.

$\mathcal{M}, w \Vdash_v \Box Ix[A, x = c]$: the interpretation of A is the singleton $\{I(c)\}$ in every accessible world (if constants are rigid).

$\mathcal{M}, w \Vdash_v \Box Ix[A, \top]$: the interpretation of A is a singleton set in every accessible world.

Note: $A_c^x \rightarrow \Box A_c^x, \Box Ix[A, \top] \vdash \Box Ix[A, x = c]$.

Rigidity. II

Let's say that if $\mathcal{M}, w \Vdash_v Ix[A, \Box Iy[A, x = y]]$, then A is rigidly unique at w .

A version of the necessity of identity follows:

$$Ix[A, \Box Iy[A, x = y]], Ix[A, x = c] \vdash \Box Ix[A, x = c]$$

If A is rigidly unique, then if the A is c , this is necessarily so.

Rigid uniqueness suffices for a version of the necessity of difference:

$$Ix[A, \Box Iy[A, x = y]], Ix[A, x \neq c] \vdash \Box Ix[A, x \neq c]$$

If A is rigidly unique, then if the A is different from c , this is necessarily so.

When A is uniquely rigid, scope distinctions do not matter:

$$Ix[A, \Box Iy[A, x = y]] \vdash Ix[A, \Box B] \leftrightarrow \Box Ix[A, B]$$

Thank you to you and the EU.

Funded by the European Union (ERC, ExtenDD, project number: 101054714). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.