Russellian Logic of Definite Descriptions and Anselm's God

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Ontological Argument

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7. Comments on the logic required to validate AA.

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Several critics: Gaunilon, Kant, Russell

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But we focus on Proslogion II which is more dependent on the logic of DD than on modal logics.
Some approaches to Proslogion II in this style:
Barnes, Oppenheimer and Zalta (OZ formalisation).

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1. Thus even the fool is convinced that something than which nothing greater can be conceived is in the understanding, since when he hears this, he understands it; and whatever is understood is in the understanding. (II.8)
2. And certainly that than which a greater cannot be conceived cannot be in the understanding alone. (II.9)
3. For if it is even in the understanding alone, it can be conceived to exist in reality also, which is greater. (II.10)
4. Thus if that than which a greater cannot be conceived is in the understanding alone, then that than which a greater cannot be conceived is itself that than which a greater can be conceived. (II.11)
5. But surely this cannot be. (II.12)
6. Thus without doubt something than which a greater cannot be conceived exists, both in the understanding and in reality. (II.13)

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OZ choose (b), I prefer (a) since it helps to treat ‘is conceived’ and ‘is in the understanding’ as synonyms and, in accordance with Anselm’s conviction, makes the presence of the concept of God in the fool’s understanding not something which is just possible but which simply holds – ‘Thus even the fool is convinced...’

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4. 'conceived' as applied to (a) objects or to (b) states;
Barnes prefers (b) but using DD helps to avoid this problem and takes (a) as the only one required.

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3. Barnes is using $\imath x\neg\exists y yI : Gzx$, where $xI : \varphi = x$ can imagine (conceive) that φ (paramodal binary operator)

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There are modal notions (we treat 'certainly' as expressing necessity of this statement) but not essential; modal operators in the prefix have rather a rhetoric value; they serve to emphasize that what is expressed in their scope is not possible. After using modal principle T (the medieval principle *ab necesse ad esse valet consequentia*), the interdefinability of $\Box$ and $\Diamond$ and propositional logic, we obtain simply $A \rightarrow B$.

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As for B it introduces implicitly the notion of existence in reality as opposed to being in the understanding; in II.13 it is used 'exists' explicitly. Thus it may be formalised either as Ex or $\exists xRx$ or $\exists xEx$.

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I follow interpretation 1. because such a formalisation of II.10 is closer to Anselm's formulation. In particular it seems that II.11 is based on such interpretation and does not make sense if we assume that the thing greater than $\imath x \varphi(x)$ is something else.

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On the other hand, its occurrence in AA supports rather interpretation 1 of II.10.

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On this basis we obtain by modus tollens $E\imath x\varphi$, hence the conclusion in sentence II.13 follows which is $A \wedge B$.

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Therefore the crucial thing is to show that C (on whaterver interpretation) leads to contradiction, and this step requires the application of proper logic of DD.

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1.	A	(sentence II.8, premiss 1)
Show:	$\Box\neg\Diamond(A \wedge \neg B)$	(sentence II.9, the goal)
2.	$A \wedge \neg B \rightarrow C$	(sentence II.10, premiss 2)
2.1.	$A \wedge \neg B$	assumption
2.2.	C	2,2.1
2.3.	D	(supposed to follow from 2.2.)
3.	$A \wedge \neg B \rightarrow D$	(sentence II.11, by 2.1–2.3)
4.	$D \rightarrow \bot$	(sentence II.12)
5.	$\neg(A \wedge \neg B)$	3,4
6.	B	1,5
7.	$A \wedge B$	(sentence II.13) 1,6

where A states that God (i.e. respective DD) is in the understanding, B that it does not exist in reality, C that 'it can be conceived to exist in reality also, which is greater', D states the identity of two contradictory descriptions.

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f - a fool

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Mxy = y is in the understanding (mind) of x

Gxy = x is greater than y

Ex = x exists in reality

$xI : \varphi$ = x can imagine (conceive) that φ

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Note that although he is denying the modal character of AA, he in fact add some binary (para-modal) operator I :.
Moreover he is adding it in two forms which seems to suggest that he intends to make a distinction between 'conceiving x' and 'conceiving that φ; but it is not clear.

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Barnes' God's description is $\imath x\varphi(x)$, where $\varphi(x) := \neg\exists yyI : Gzx$
It seems to be mistaken and should be rather: $\varphi(x) := \neg\exists zyI : Gzx$

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There are 5 premisses:

1.	$Uf\imath x\varphi$	Pr 1
2.	$\forall xy(Uxy \rightarrow Mxy)$	Pr 2
3.	$\forall xy(Mxy \rightarrow xI : Ey)$	Pr 3
4.	$\forall y(\exists xMxy \wedge \neg Ey \rightarrow \forall z(Ez \rightarrow Gzy))$	Pr 4 [II.10]
5.	$\forall \varphi\psi((\varphi \rightarrow \psi) \rightarrow \forall x(xI : \varphi \rightarrow xI : \psi))$	Pr 5

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5.	$\forall \varphi\psi((\varphi \rightarrow \psi) \rightarrow \forall x(xI : \varphi \rightarrow xI : \psi))$	Pr 5

Note that Pr 5 is a postulate concerning I :, moreover it is in the second-order logic [additional ad hoc enrichment].

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4.	$\forall y(\exists xMxy \wedge \neg Ey \rightarrow \forall z(Ez \rightarrow Gzy))$	Pr 4
5.	$\forall \varphi\psi((\varphi \rightarrow \psi) \rightarrow \forall x(xI : \varphi \rightarrow xI : \psi))$	Pr 5
6.	$\neg E\imath x\varphi$	indirect ass.
7.	$Uf\imath x\varphi \rightarrow Mf\imath x\varphi$	2
8.	$Mf\imath x\varphi \rightarrow fI : E\imath x\varphi$	3
9.	$Mf\imath x\varphi$	1,7
10.	$fI : E\imath x\varphi$	8,9
11.	$\exists xMf\imath x\varphi \wedge \neg E\imath x\varphi \rightarrow \forall z(Ez \rightarrow Gz\imath x\varphi)$	4
12.	$\exists xMx\imath x\varphi$	9
13.	$\exists xMx\imath x\varphi \wedge \neg E\imath x\varphi$	6,12
14.	$\forall z(Ez \rightarrow Gz\imath x\varphi)$	11,13
15.	$E\imath x\varphi \rightarrow G\imath x\varphi\imath x\varphi$	14
16.	$(E\imath x\varphi \rightarrow G\imath x\varphi\imath x\varphi) \rightarrow \forall x(xI : E\imath x\varphi \rightarrow xI : G\imath x\varphi\imath x\varphi)$	5
17.	$\forall x(xI : E\imath x\varphi \rightarrow xI : G\imath x\varphi\imath x\varphi)$	15,16
18.	$fI : E\imath x\varphi \rightarrow fI : G\imath x\varphi\imath x\varphi$	17
19.	$fI : G\imath x\varphi\imath x\varphi$	10,18
20.	$fI : Gz\imath x\varphi$	19, RImag
21.	$\exists yyI : Gz\imath x\varphi := \exists yyI : Gz\imath x\neg\exists yyI : Gzx$	20
22.	$\perp$	21

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Also his rule RImag is not necessary if we correct God's description.

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RA $\psi(\imath x\varphi(x)) \leftrightarrow \exists y(\forall x(\varphi(x) \leftrightarrow x = y) \wedge \psi(y))$

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The general ones derivable from Russell axiom:

Description theorem 1: $\exists_1 x\varphi \rightarrow \exists y(y = \imath x\varphi)$
Lemma 1: $t = \imath x\varphi \rightarrow \varphi[x/t]$
Description theorem 2: $\exists y(y = \imath x\varphi) \rightarrow \varphi[x/\imath x\varphi]$

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Specific assumption:

Connectedness of G: $\forall xy(Gxy \vee Gyx \vee x = y)$
Lemma 2: $\exists x \varphi_1 \rightarrow \exists_1 x \varphi_1$

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Premiss 1 $\exists x \varphi_1$ [II.8]
Premiss 2 $\neg E \imath x \varphi_1 \rightarrow \exists y (G y \imath x \varphi_1 \wedge C y)$ [II.10]
Conclusion $E \imath x \varphi_1$ [II.13]

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Some other formulations (in response to objections) of Premiss 2 were considered:
Premiss 2' $\forall x (\neg Ex \rightarrow \exists y (Gyx \wedge Cy))$
Premiss 2" $\forall x (\varphi_1 \wedge \neg Ex \rightarrow \exists y (Gyx \wedge Cy))$
Premiss 2"' $\forall x (Cx \wedge \neg Ex \rightarrow \exists y (Gyx \wedge Cy))$ [Kalish and Montague]

OZ formalisation I:

The Proof:

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The Proof:

1.	$\exists x \varphi_1$	Pr 1
2.	$\exists_1 x \varphi_1$	1, Lemma 2
3.	$\exists y(y = \imath x \varphi_1)$	2, Description theorem 1
4.	$C \imath x \varphi_1 \wedge \neg \exists y(G y \imath x \varphi_1 \wedge C y)$	3, Description theorem 2
5.	$\neg \exists y(G y \imath x \varphi_1 \wedge C y)$	4
6.	$\neg E \imath x \varphi_1 \rightarrow \exists y(G y \imath x \varphi_1 \wedge C y)$	Pr 2
7.	$E \imath x \varphi_1$	5, 6

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6.	$\neg E\imath x \varphi_1 \rightarrow \exists y(Gy\imath x \varphi_1 \wedge Cy)$	Pr 2
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It holds also if we change Premiss 2 for other variant considered above.

OZ formalisation II:

Simplified Approach discovered by PROVER9:

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In the second version they use only:

Prem 2: $\neg E\imath x\varphi_1 \rightarrow \exists y(Gy\imath x\varphi_1 \wedge Cy)$ [II.10]
Description theorem 2: $\exists y(y = \imath x\varphi) \rightarrow \varphi[x/\imath x\varphi]$

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Prem 2: $\neg E\imath x\varphi_1 \rightarrow \exists y(Gy\imath x\varphi_1 \wedge Cy)$ [II.10]
Description theorem 2: $\exists y(y = \imath x\varphi) \rightarrow \varphi[x/\imath x\varphi]$

and additionally

Description theorem 3: $\psi[z/\imath x\varphi] \rightarrow \exists y(y = \imath x\varphi)$, i.e. SA

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1.	$\neg E\imath x\varphi_1$	indirect assumption
2.	$\neg E\imath x\varphi_1 \rightarrow \exists y(Gy\imath x\varphi_1 \wedge Cy)$	Pr 2
3.	$\exists y(Gy\imath x\varphi_1 \wedge Cy)$	1,2
3.1.	$Ga\imath x\varphi_1 \wedge Ca$	3, assumption with fresh a
3.2.	$Ga\imath x\varphi_1$	3.1.
3.3.	$\exists y(y = \imath x\varphi_1)$	3.2, Description theorem 3
4.	$\exists y(y = \imath x\varphi_1)$	3, 3.1–3.3, by discharge
5.	$C\imath x\varphi_1 \wedge \neg\exists y(Gy\imath x\varphi_1 \wedge Cy)$	4, Description theorem 2
6.	$\neg\exists y(Gy\imath x\varphi_1 \wedge Cy)$	5
7.	$E\imath x\varphi_1$	from contradiction 3, 6

OZ formalisation II:

Alernative proof in SC for RDD:

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$$\dfrac{Gab \Rightarrow Gab \qquad Ca \Rightarrow Ca}{Gab, Ca \Rightarrow Gab \wedge Ca}\ (\Rightarrow \wedge)$$

$$\dfrac{Gab, Ca \Rightarrow Gab \wedge Ca}{Gab, Ca \Rightarrow \exists y(Gyb \wedge Cy)}\ (\Rightarrow \exists)$$

$$\dfrac{Gab, Ca \Rightarrow \exists y(Gyb \wedge Cy)}{\neg\exists y(Gyb \wedge Cy), Gab, Ca \Rightarrow}\ (\neg \Rightarrow)$$

$$\dfrac{\neg\exists y(Gyb \wedge Cy), Gab, Ca \Rightarrow}{Cb, \neg\exists y(Gyb \wedge Cy), Gab, Ca \Rightarrow}\ (W \Rightarrow)$$

$$\dfrac{Cb, \neg\exists y(Gyb \wedge Cy), Gab, Ca \Rightarrow}{Cb \wedge \neg\exists y(Gyb \wedge Cy), Gab, Ca \Rightarrow}\ (\wedge \Rightarrow)$$

$$\dfrac{\Rightarrow b = b \qquad Cb \wedge \neg\exists y(Gyb \wedge Cy), Gab, Ca \Rightarrow}{b = \imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)), Gab, Ca \Rightarrow}\ (\imath \Rightarrow)$$

$$\dfrac{b = \imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)), Gab, Ca \Rightarrow}{Ga\imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)), Ca \Rightarrow}\ (Str \Rightarrow)$$

$$\dfrac{Ga\imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)), Ca \Rightarrow}{Ga\imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)) \wedge Ca \Rightarrow}\ (\wedge \Rightarrow)$$

$$\dfrac{Ga\imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)) \wedge Ca \Rightarrow}{\exists y(Gy\imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)) \wedge Cy) \Rightarrow}\ (\exists \Rightarrow)$$

$$\dfrac{\exists y(Gy\imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)) \wedge Cy) \Rightarrow}{\Rightarrow \neg\exists y(Gy\imath x(Cx \wedge \neg\exists y(Gyx \wedge Cy)) \wedge Cy)}\ (\Rightarrow \neg)$$

the only required elements are Prem 2, $(Str \Rightarrow)$ corresponding to DesTh 3 and $(\imath \Rightarrow)$ corresponding to one direction of RA (or rather LA).

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4. no distinction between 'conceived t' and 'conceived that φ'.

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II.8	$C\imath x\varphi_2$
II.9	$\Box\neg\Diamond(C\imath x\varphi_2 \wedge \neg E\imath x\varphi_2)$
II.10	$C\imath x\varphi_2 \wedge \neg E\imath x\varphi_2 \rightarrow$ $\Diamond C\imath z(Ez \wedge z = \imath x\varphi_2) \wedge G\imath z(Ez \wedge z = \imath x\varphi_2)\imath x\varphi_2$
II.11	$C\imath x\varphi_2 \wedge \neg E\imath x\varphi_2 \rightarrow \imath x\varphi_2 = \imath x\exists y(Gyx \wedge \Diamond Cy)$
II.12	$\imath x\varphi_2 = \imath x\exists y(Gyx \wedge \Diamond Cy) \rightarrow \bot$
II.13	$C\imath x\varphi_2 \wedge E\imath x\varphi_2$

where $\imath x\varphi_2 := \imath x\neg\exists y(Gyx \wedge \Diamond Cy)$

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The proof of the crucial part:

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1.	$\neg E\imath x\varphi_2$	indirect ass.
2.	$C\imath x\varphi_2$	Pr 1
3.	$C\imath x\varphi_2 \wedge \neg E\imath x\varphi_2 \rightarrow$	
	$\Diamond C\imath z(Ez \wedge z = \imath x\varphi_2) \wedge G\imath z(Ez \wedge z = \imath x\varphi_2)\imath x\varphi_2$	Pr 2
4.	$\Diamond C\imath z(Ez \wedge z = \imath x\varphi_2) \wedge G\imath z(Ez \wedge z = \imath x\varphi_2)\imath x\varphi_2$	1, 2, 3
5.	$G\imath z(Ez \wedge z = \imath x\varphi_2)\imath x\varphi_2$	4
6.	$\exists y(\forall z(Ez \wedge z = \imath x\varphi_2 \leftrightarrow z = y) \wedge Gy\imath x\varphi_2)$	5, $RA^{\rightarrow}$
6.1.	$\forall z(Ez \wedge z = \imath x\varphi_2 \leftrightarrow z = a) \wedge Ga\imath x\varphi_2$	6, existential ass.
6.2.	$\forall z(Ez \wedge z = \imath x\varphi_2 \leftrightarrow z = a)$	6.1
6.3.	$Ea \wedge a = \imath x\varphi_2 \leftrightarrow a = a$	6.2
6.4.	Ea	6.3
6.5.	$a = \imath x\varphi_2$	6.3
6.6.	$E\imath x\varphi_2$	6.4, 6.5
7.	$E\imath x\varphi_2$	6.1–6.6, by discharge of 6.1

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In fact the proof goes even if we use the following premiss 2:

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It seems that to validate OZ we need full strength of RA, whereas for this proof of AA $RA^{\rightarrow}$ is enough. But ...

OZ II again:

Another proof of OZ [only $RA^{\rightarrow}$ used]:

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1.	$\neg E\imath x\varphi_1$	indirect assumption
2.	$\neg E\imath x\varphi_1 \rightarrow \exists y(Gy\imath x\varphi_1 \wedge Cy)$	Pr 2
3.	$\exists y(Gy\imath x\varphi_1 \wedge Cy)$	1,2
3.1.	$Ga\imath x\varphi_1 \wedge Ca$	3, existential ass. with fresh a
3.2.	$Ga\imath x\varphi_1$	3.1
3.3.	Ca	3.1
3.4.	$\exists y(\forall x(\varphi_1 \leftrightarrow x = y) \wedge Gay)$	3.2, $RA^{\rightarrow}$
3.4.1.	$\forall x(\varphi_1 \leftrightarrow x = b) \wedge Gab$	3.4, existential ass. with fresh b
3.4.2.	$(\varphi_1[x/b] \leftrightarrow b = b) \wedge Gab$	3.4.1
3.4.3.	$\varphi_1[x/b] \leftrightarrow b = b$	3.4.2
3.4.4.	Gab	3.4.2
3.4.5.	$Cb \wedge \neg\exists y(Gyb \wedge Cy)$	3.4.3 [:= $\varphi_1[x/b]$]
3.4.6.	$\neg\exists y(Gyb \wedge Cy)$	3.4.5
3.4.7.	$Gab \wedge Ca$	3.4.4, 3.3
3.4.8.	$\exists y(Gyb \wedge Cy)$	3.4.7
3.4.9.	$\bot$	3.4.6, 3.4.8
3.5.	$\bot$	3.4.1-3.4.9, by discharge of 3.4.1
4.	$\bot$	3.1-3.5, by discharge of 3.1

The Weakest Logic validating AA:

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Let us analyse OZ again:

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3.	$\exists y(Gy\imath x\varphi_1 \wedge Cy)$	1,2
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3.2.	$Ga\imath x\varphi_1$	3.1
3.3.	Ca	3.1
3.4.	$\exists x\varphi_1$	3.2, RAa
3.4.1.	$Cb \wedge \neg\exists z(Gzb \wedge Cz)$	3.4, existential ass. with fresh b
3.4.2.	$\forall x(\varphi_1 \rightarrow Gax)$	3.2, RAb
3.4.3.	$Cb \wedge \neg\exists z(Gzb \wedge Cz) \rightarrow Gab$	3.4.2
3.4.4.	Gab	3.4.1, 3.4.3
3.4.5.	$Gab \wedge Ca$	3.4.4, 3.3
3.4.6.	$\exists y(Gyb \wedge Cy)$	3.4.7
3.4.7.	$\neg\exists y(Gyb \wedge Cy)$	3.4.1
3.4.8.	$\perp$	3.4.6, 3.4.7
3.5.	$\perp$	3.4.1-3.4.8, by discharge of 3.4.1
4.	$\perp$	3.1-3.5, by discharge of 3.1

The Weakest Logic validating AA:

Another proof of OZ [only RAa, RAb used]:

1.	$\neg E\imath x\varphi_1$	indirect assumption
2.	$\neg E\imath x\varphi_1 \to \exists y(Gy\imath x\varphi_1 \wedge Cy)$	Pr 2
3.	$\exists y(Gy\imath x\varphi_1 \wedge Cy)$	1,2
3.1.	$Ga\imath x\varphi_1 \wedge Ca$	3, existential ass. with fresh a
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3.4.5.	$Gab \wedge Ca$	3.4.4, 3.3
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3.4.7.	$\neg\exists y(Gyb \wedge Cy)$	3.4.1
3.4.8.	$\bot$	3.4.6, 3.4.7
3.5.	$\bot$	3.4.1-3.4.8, by discharge of 3.4.1
4.	$\bot$	3.1-3.5, by discharge of 3.1

only RAa and RAb is needed.

Another proof of my formalisation:

The Weakest Logic validating AA:

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1.	$\neg E\imath x\varphi_2$	ind. ass.
2.	$C\imath x\varphi_2$	Pr 1
3.	$C\imath x\varphi_2 \wedge \neg E\imath x\varphi_2 \rightarrow$	
	$\Diamond C\imath z(Ez \wedge z = \imath x\varphi_2) \wedge G\imath z(Ez \wedge z = \imath x\varphi_2)\imath x\varphi_2$	Pr 2
4.	$\Diamond C\imath z(Ez \wedge z = \imath x\varphi_2) \wedge G\imath z(Ez \wedge z = \imath x\varphi_2)\imath x\varphi_2$	1, 2, 3
5.	$G\imath z(Ez \wedge z = \imath x\varphi_2)\imath x\varphi_2$	4
6.	$\exists z(Ez \wedge z = \imath x\varphi_2)$	5, RAa
6.1.	$Ea \wedge a = \imath x\varphi_2$	6, ex. ass.
6.2.	Ea	6.1
6.3.	$a = \imath x\varphi_2$	6.2
6.4.	$E\imath x\varphi_2$	6.2, 6.3
7.	$E\imath x\varphi_2$	6.1–6.4

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Only RAa is necessary! So objectively my formalisation requires even weaker logic than OZ.

Other logics validating AA? – The role of premiss 1

The Weakest Logic validating AA:

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However, note that in the first formalisation of OZ $\exists_1 x\varphi$ follows from $\exists x\varphi$ by connectedness of G.

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