Approaches to Formalisation of the Logic of Classes and its Extension to ZFC

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ExtenDD seminar April, 2026

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3. Sequent calculi for logics of classes.
4. Extension of the language.
5. Sequent calculus for serious set theory (ZFC as an example).

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Classes represented by set-abstracts/sets (= existing classes) represented by variables.

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We focus on VTC and provide its proof-theoretic characterisation GVTC in two variants for classic version.
Then we provide a sequent calculus for free Scott's variant.

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t for any terms;
s for any set abstracts.

$a \in b, a = b$ – simple atoms.
$s \in t, t \in s, s = t$ – complex atoms.
$t \approx t'$ for $t = t'$ or $t' = t$
φ, ψ – formulae;
Γ, Δ – multisets of formulae;
$\varphi[x/t]$ – proper substitution of t for x in φ.

Quine's VTC

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For quantifiers it is classical with respect to variables (it is pure FOL in the sense of Church) but free with respect to set abstract, i.e. instantiation of s is restricted as follows:

$$s \in U \wedge \forall x \varphi \rightarrow \varphi[x/s] \quad (1)$$

$$s \in U \wedge \varphi[x/s] \rightarrow \exists x \varphi \quad (2)$$

where $U := \{x : x = x\}$

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where $U := \{x : x = x\}$

Note that existence for classes (as expressed by set abstracts) can be expressed in three equivalent forms:

1. $\{x : \varphi\} \in U$
2. $\exists y(y = \{x : \varphi\})$
3. $\exists y \forall x(x \in y \leftrightarrow \varphi)$

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For identity VTC satisfies for arbitrary terms the Leibniz Law and Extensionality:

$$t_1 = t_2 \wedge \varphi[x/t_1] \rightarrow \varphi[x/t_2] \tag{LL}$$

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These principles imply usual properties of reflexivity, symmetry, transitivity for arbitrary terms and *AV* (α-conversion) for set abstracts:

$$\{x : \varphi\} = \{y : \varphi[x/y]\} \tag{AV}$$

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We can replace them with Scott's axioms:

$$\forall y(y \in \{x : \varphi\} \leftrightarrow \exists x(x = y \wedge \varphi)) \tag{Q3}$$

$$s \in t \rightarrow \exists y(y = s) \tag{Q2}$$

where y is not free in s, t, φ.

SEQUENT CALCULUS G FOR FOL

$$(Cut)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (AX)\ \varphi \Rightarrow \varphi$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \qquad (W\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad (\wedge\Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow W)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\vee\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \qquad (C\Rightarrow)\ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \qquad (\Rightarrow C)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\leftrightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi \qquad \varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \leftrightarrow \psi, \Gamma \Rightarrow \Delta} \qquad (\forall\Rightarrow)\ \frac{\varphi[x/b], \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/b]}{\Gamma \Rightarrow \Delta, \exists x\varphi}$$

$$(\Rightarrow\leftrightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \leftrightarrow \psi} \qquad (\Rightarrow\forall)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi} \qquad (\exists\Rightarrow)\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta}$$

where a is a fresh parameter (eigenvariable)

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HOW TO BUILD SEQUENT CALCULUS FOR VTC?

Two approaches to construction of the rules for VTC:

- Tennant: rules for single-barelled abstraction principles characterising $b = \{x : \varphi\} \Longrightarrow$ T-GVTC;
- rules constructed on the basis of extensionality principle characterising $t = t'$ generally $\Longrightarrow$ GVTC.

SPECIAL RULES FOR T-GVTC:

$(L1)\ \dfrac{\Gamma\Rightarrow\Delta, t=b \qquad \Gamma\Rightarrow\Delta, t=c}{\Gamma\Rightarrow\Delta, b=c}$ $\qquad (L2)\ \dfrac{\Gamma\Rightarrow\Delta, a=b \qquad \Gamma\Rightarrow\Delta, a\in c}{\Gamma\Rightarrow\Delta, b\in c}$

$(L3)\ \dfrac{\Gamma\Rightarrow\Delta, a=b \qquad \Gamma\Rightarrow\Delta, c\in a}{\Gamma\Rightarrow\Delta, c\in b}$ $\qquad (\Rightarrow:)\ \dfrac{t[a],\Gamma\Rightarrow\Delta, \varphi[x/a] \qquad \varphi[x/a],\Gamma\Rightarrow\Delta, t[a]}{\Gamma\Rightarrow\Delta, t\approx\{x:\varphi\}}$

$(:\Rightarrow 1)\ \dfrac{\Gamma\Rightarrow\Delta, t[b] \qquad \varphi[x/b],\Gamma\Rightarrow\Delta}{t\approx\{x:\varphi\},\Gamma\Rightarrow\Delta}$ $\qquad (:\Rightarrow 2)\ \dfrac{\Gamma\Rightarrow\Delta, \varphi[x/b] \qquad t[b],\Gamma\Rightarrow\Delta}{t\approx\{x:\varphi\},\Gamma\Rightarrow\Delta}$

$(\in\Rightarrow 1)\ \dfrac{\varphi[x/b],\Gamma\Rightarrow\Delta}{b\in\{x:\varphi\},\Gamma\Rightarrow\Delta}$ $\qquad (\Rightarrow\in 1)\ \dfrac{\Gamma\Rightarrow\Delta, \varphi[x/b]}{\Gamma\Rightarrow\Delta, b\in\{x:\varphi\}}$

$(\in\Rightarrow 2)\ \dfrac{a=\{x:\varphi\}, a\in t,\Gamma\Rightarrow\Delta}{\{x:\varphi\}\in t,\Gamma\Rightarrow\Delta}$ $\qquad (\Rightarrow\in 2)\ \dfrac{\Gamma\Rightarrow\Delta, b=\{x:\varphi\} \qquad \Gamma\Rightarrow\Delta, b\in t}{\Gamma\Rightarrow\Delta, \{x:\varphi\}\in t}$

where: a is fresh in $(\Rightarrow:)$, $(\in\Rightarrow 2)$, $t[b]$ is either $b\in t$, if t is a parameter or $\psi[v/b]$, if $t := \{v:\psi\}$ for v not necessarily identical to x.

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$(\in \Rightarrow 2)\ \dfrac{a = \{x : \varphi\}, a \in t, \Gamma \Rightarrow \Delta}{\{x : \varphi\} \in t, \Gamma \Rightarrow \Delta}$ $\qquad (\Rightarrow \in 2)\ \dfrac{\Gamma \Rightarrow \Delta, b = \{x : \varphi\} \qquad \Gamma \Rightarrow \Delta, b \in t}{\Gamma \Rightarrow \Delta, \{x : \varphi\} \in t}$

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Warning: $(L1)$ does not satisfy the subformula property.

SPECIAL RULES FOR GVTC:

$$(\in\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, b_1 = b_2 \qquad b_2 \in b_3, \Gamma \Rightarrow \Delta}{b_1 \in b_3, \Gamma \Rightarrow \Delta} \qquad (\in\Rightarrow 1)\ \frac{\varphi[x/b], \Gamma \Rightarrow \Delta}{b \in \{x : \varphi\}, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow=)\ \frac{a \in t, \Gamma \Rightarrow \Delta, a \in t' \qquad a \in t', \Gamma \Rightarrow \Delta, a \in t}{\Gamma \Rightarrow \Delta, t = t'} \qquad (\Rightarrow\in 1)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/b]}{\Gamma \Rightarrow \Delta, b \in \{x : \varphi\}}$$

$$(=\Rightarrow 1)\ \frac{\Gamma \Rightarrow \Delta, b \in t \qquad b \in t', \Gamma \Rightarrow \Delta}{t = t', \Gamma \Rightarrow \Delta} \qquad (=\Rightarrow 2)\ \frac{\Gamma \Rightarrow \Delta, b \in t' \qquad b \in t, \Gamma \Rightarrow \Delta}{t = t', \Gamma \Rightarrow \Delta}$$

$$(\in\Rightarrow 2)\ \frac{a = \{x : \varphi\}, a \in t, \Gamma \Rightarrow \Delta}{\{x : \varphi\} \in t, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\in 2)\ \frac{\Gamma \Rightarrow \Delta, b = \{x : \varphi\} \qquad \Gamma \Rightarrow \Delta, b \in t}{\Gamma \Rightarrow \Delta, \{x : \varphi\} \in t}$$

where: a is fresh in $(\Rightarrow=), (\in\Rightarrow 2)$.

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9. Cut-free system for T-GVTC is not fully analytic due to 2.
10. Both systems prove consistency of VTC, due to 3 and 7.
11. Hypothesis: The system T-GVTC may be proved to satisfy the subformula property.

EXTENSIONS

$\bigcup t := \{x : \exists y(x \in y \wedge y \in t)\}.$
$t \cap t' := \{x : x \in t \wedge x \in t'\}.$
$t \cup t' := \{x : x \in t \vee x \in t'\}.$
$t \setminus t' := \{x : x \in t \wedge x \notin t'\}.$
$t \subseteq t' \Leftrightarrow \forall x(x \in t \rightarrow x \in t').$
$\mathcal{P}(t) := \{x : x \subseteq t\}.$
$\{t, t'\} := \{x : x = t \vee x = t'\}.$
$\langle t, t' \rangle := \{x : x = \{t\} \vee x = \{t, t'\}\}.$
$t \times t' := \{x : \exists y, z(x = \langle y, z \rangle \wedge y \in t \wedge z \in t')\}.$
$REL(t, t') := \{x : x \subseteq t \times t'\}.$
$DOM(t) := \{x : \exists y(\langle x, y \rangle \in t)\}.$
$UN(t, t') := \{x : \forall y \in t, \forall z, v \in t'(\langle y, z \rangle \in x \ \wedge \langle y, v \rangle \in x \rightarrow z = v)\}$
$FUN(t, t') := \{x : x \in REL(t, t') \wedge DOM(x) = t \ \wedge x \in UN(t, t')\}.$
$CON(t) \Leftrightarrow \forall x, y \in t(x \in y \vee y \in x \vee x = y).$
$TR(t) \Leftrightarrow \forall x(x \in t \rightarrow x \subseteq t).$
$ON(t) \Leftrightarrow CON(t) \wedge TR(t).$

EXTENSIONS

$$(\emptyset\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta, c=c}{c\in\emptyset,\Gamma\Rightarrow\Delta}\quad(\Rightarrow\emptyset)\ \frac{c=c,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta, c\in\emptyset}\quad(U\Rightarrow)\ \frac{c=c,\Gamma\Rightarrow\Delta}{c\in U,\Gamma\Rightarrow\Delta}\quad(\Rightarrow U)\ \frac{\Gamma\Rightarrow\Delta, c=c}{\Gamma\Rightarrow\Delta, c\in U}$$

$$(\cap\Rightarrow)\ \frac{c\in t, c\in t',\Gamma\Rightarrow\Delta}{c\in t\cap t',\Gamma\Rightarrow\Delta}\quad(\Rightarrow\cap)\ \frac{\Gamma\Rightarrow\Delta, c\in t\quad \Gamma\Rightarrow\Delta, c\in t'}{\Gamma\Rightarrow\Delta, c\in t\cap t'}\quad(\cup\Rightarrow)\ \frac{c\in t,\Gamma\Rightarrow\Delta\quad c\in t',\Gamma\Rightarrow\Delta}{c\in t\cup t',\Gamma\Rightarrow\Delta}$$

$$(\Rightarrow\cup)\ \frac{\Gamma\Rightarrow\Delta, c\in t, c\in t'}{\Gamma\Rightarrow\Delta, c\in t\cup t'}\quad(\setminus\Rightarrow)\ \frac{c\in t,\Gamma\Rightarrow\Delta, c\in t'}{c\in t\setminus t',\Gamma\Rightarrow\Delta}\quad(\Rightarrow\setminus)\ \frac{\Gamma\Rightarrow\Delta, c\in t\qquad c\in t',\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta, c\in t\setminus t'}$$

$$(\subseteq\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta, b\in t\quad b\in t',\Gamma\Rightarrow\Delta}{t\subseteq t',\Gamma\Rightarrow\Delta}\quad(\Rightarrow\subseteq)\ \frac{a\in t,\Gamma\Rightarrow\Delta, a\in t'}{\Gamma\Rightarrow\Delta, t\subseteq t'}\quad(\mathcal{P}\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta, b\in c\quad b\in t,\Gamma\Rightarrow\Delta}{c\in\mathcal{P}(t),\Gamma\Rightarrow\Delta}$$

$$(\Rightarrow\mathcal{P})\ \frac{a\in c,\Gamma\Rightarrow\Delta, a\in t}{\Gamma\Rightarrow\Delta, c\in\mathcal{P}(t)}\quad(\{\}\Rightarrow)\ \frac{c=t,\Gamma\Rightarrow\Delta\qquad c=t',\Gamma\Rightarrow\Delta}{c\in\{t,t'\},\Gamma\Rightarrow\Delta}\quad(\Rightarrow\{\})\ \frac{\Gamma\Rightarrow\Delta, c=t, c=t'}{\Gamma\Rightarrow\Delta, c\in\{t,t'\}}$$

$$(\bigcup\Rightarrow)\ \frac{c\in a, a\in t,\Gamma\Rightarrow\Delta}{c\in\bigcup t,\Gamma\Rightarrow\Delta}\quad(\Rightarrow\bigcup)\ \frac{\Gamma\Rightarrow\Delta, c\in b\qquad \Gamma\Rightarrow\Delta, b\in t}{\Gamma\Rightarrow\Delta, c\in\bigcup t}$$

where: a is fresh, whereas b, c are arbitrary parameters.

EXTENSIONS

$$(\langle\rangle \Rightarrow)\ \frac{c = \{t\}, \Gamma \Rightarrow \Delta \qquad c = \{t, t'\}, \Gamma \Rightarrow \Delta}{c \in \langle t, t' \rangle, \Gamma \Rightarrow \Delta} \quad (\Rightarrow \langle\rangle)\ \frac{\Gamma \Rightarrow \Delta, c = \{t\}, c = \{t, t'\}}{\Gamma \Rightarrow \Delta, c \in \langle t, t' \rangle}$$

$$(\times \Rightarrow)\ \frac{c = \langle a, a' \rangle, a \in t, a' \in t', \Gamma \Rightarrow \Delta}{c \in t \times t', \Gamma \Rightarrow \Delta} (\Rightarrow \times)\ \frac{\Gamma \Rightarrow \Delta, c = \langle b, b' \rangle \quad \Gamma \Rightarrow \Delta, b \in t \quad \Gamma \Rightarrow \Delta, b' \in t'}{\Gamma \Rightarrow \Delta, c \in t \times t'}$$

$$(REL \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, b \in c \qquad b \in t \times t', \Gamma \Rightarrow \Delta}{c \in REL(t, t'), \Gamma \Rightarrow \Delta} \quad (\Rightarrow REL)\ \frac{a \in c, \Gamma \Rightarrow \Delta, c \in t \times t'}{\Gamma \Rightarrow \Delta, c \in REL(t, t')}$$

$$(DOM \Rightarrow)\ \frac{\langle c, a \rangle \in t, \Gamma \Rightarrow \Delta}{c \in DOM(t), \Gamma \Rightarrow \Delta} \quad (\Rightarrow DOM)\ \frac{\Gamma \Rightarrow \Delta, \langle c, b \rangle \in t}{\Gamma \Rightarrow \Delta, c \in DOM(t)}$$

$$(\Rightarrow UN)\ \frac{a \in t, e \in t', e' \in t', \langle a, e \rangle \in c, \langle a, e' \rangle \in c, \Gamma \Rightarrow \Delta, e = e'}{\Gamma \Rightarrow \Delta, c \in UN(t, t')}$$

$(UN \Rightarrow)$

$$\frac{\Gamma \Rightarrow \Delta, b \in t \quad \Gamma \Rightarrow \Delta, d \in t' \quad \Gamma \Rightarrow \Delta, d' \in t' \quad \Gamma \Rightarrow \Delta, \langle b, d \rangle \in c \quad \Gamma \Rightarrow \Delta, \langle b, d' \rangle \in c \quad d = d', \Gamma \Rightarrow \Delta}{c \in UN(t, t'), \Gamma \Rightarrow \Delta}$$

where: a, a', e, e' are fresh, whereas b, b', c, d, d' are arbitrary parameters.

EXTENSIONS

$$(FUN \Rightarrow) \quad \frac{c \in REL(t,t'), DOM(c) = t, c \in UN(t,t'), \Gamma \Rightarrow \Delta}{c \in FUN(t,t'), \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow FUN) \quad \frac{\Gamma \Rightarrow \Delta, c \in REL(t,t') \quad \Gamma \Rightarrow \Delta, DOM(c) = t \quad \Gamma \Rightarrow \Delta, c \in UN(t,t')}{\Gamma \Rightarrow \Delta, c \in FUN(t,t')}$$

$$(CON \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, b \in t \qquad \Gamma \Rightarrow \Delta, c \in t \qquad b \in c, \Gamma \Rightarrow \Delta \qquad c \in b, \Gamma \Rightarrow \Delta \qquad b = c, \Gamma \Rightarrow \Delta}{CON(t), \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow CON) \quad \frac{a \in t, a' \in t, \Gamma \Rightarrow \Delta, a \in a', a' \in a, a = a'}{\Gamma \Rightarrow \Delta, CON(t)}$$

$$(\Rightarrow TR) \quad \frac{a \in t, a' \in a, \Gamma \Rightarrow \Delta, a' \in t}{\Gamma \Rightarrow \Delta, TR(t)} \quad (\Rightarrow TR) \quad \frac{\Gamma \Rightarrow \Delta, b \in t \qquad \Gamma \Rightarrow \Delta, c \in b \qquad c \in t, \Gamma \Rightarrow \Delta}{TR(t), \Gamma \Rightarrow \Delta}$$

$$(ON \Rightarrow) \quad \frac{TR(t), CON(t), \Gamma \Rightarrow \Delta}{ON(t), \Gamma \Rightarrow \Delta} \quad (\Rightarrow ON) \quad \frac{\Gamma \Rightarrow \Delta, TR(t) \qquad \Gamma \Rightarrow \Delta, CON(t)}{\Gamma \Rightarrow \Delta, ON(t)}$$

where: a, a' are fresh, whereas b, c are arbitrary parameters.

WHAT WE OBTAIN?

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EXAMPLE:

Unprovability of the Russellian class $R := \{x : x \notin x\}$.

$$
(=\Rightarrow 1)\ \dfrac{
(=\Rightarrow 2)\ \dfrac{
(\Rightarrow \in 1)\ \dfrac{
(\Rightarrow \neg)\ \dfrac{a \in a \Rightarrow a \in a}{\Rightarrow a \in a, a \notin a}
}{\Rightarrow a \in a, a \in R}
\qquad a \in a \Rightarrow a \in a
}{a = R \Rightarrow a \in a}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{a \in a \Rightarrow a \in a}{\Rightarrow a \in a, a \notin a}
}{\Rightarrow a \in a, a \in R}
\qquad a \in a \Rightarrow a \in a
}{a = R \Rightarrow a \in a}
}{a \notin a, a = R \Rightarrow}\ (\neg \Rightarrow)
}{a \in R, a = R \Rightarrow}\ (\in\Rightarrow 1)
}{}
}{
(C \Rightarrow), (W \Rightarrow)\ \dfrac{a = R, a = R \Rightarrow}{
(\in\Rightarrow 2)\ \dfrac{a = R, a \in U \Rightarrow}{
(\Rightarrow \neg)\ \dfrac{R \in U \Rightarrow}{\Rightarrow R \notin U}
}
}
}
$$

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

Axioms of Separation, Pair, Sum, Power Set:

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

Axioms of Separation, Pair, Sum, Power Set:

Separation Schema: $\forall y \exists z \forall x (x \in z \leftrightarrow x \in y \wedge \varphi(x))$, where z is not in $\varphi(x)$
Pairing Axiom: $\forall y_1 y_2 \exists z \forall x (x \in z \leftrightarrow x = y_1 \vee x = y_2)$
Union Axiom: $\forall y \exists z \forall x (x \in z \leftrightarrow \exists v (x \in v \wedge v \in y))$
Powerset Axiom: $\forall y \exists z \forall x (x \in z \leftrightarrow x \subseteq y)$

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$$(\in\Rightarrow 3)\ \frac{b \in t, \varphi[x/b], \Gamma \Rightarrow \Delta}{b \in \{x \in t : \varphi\}, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\in 3)\ \frac{\Gamma \Rightarrow \Delta, b \in t \qquad \Gamma \Rightarrow \Delta, \varphi[x/b]}{\Gamma \Rightarrow \Delta, b \in \{x \in t : \varphi\}}$$

$$(Sep)\ \frac{a = \{x \in b : \varphi(x)\}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$ where a is fresh.

There is just one rule-schema for Pairing, Union and Powerset Axioms:

$$(S)\ \frac{a = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where a is fresh and s is either $\{x : x = b_1 \vee x = b_2\}$ or $\{x : \exists y (x \in y \wedge y \in b\}$ or $\{x : x \subseteq b\}$.

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The rules for the Axiom of Infinity:

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The rules for the Axiom of Infinity:

Infinity axiom: $\exists z(\emptyset \in z \wedge \forall x(x \in z \rightarrow \{x\} \in z))$

can be characterised by means of the system of rules:

$$
(Inf2)\ \dfrac{\Gamma \Rightarrow \Delta, b \in a \qquad \{b\} \in a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}
$$

$$
\dfrac{\mathcal{D}}{\emptyset \in a, \Pi \Rightarrow \Sigma}
$$

$$
(Inf1)\ \dfrac{\emptyset \in a, \Pi \Rightarrow \Sigma}{\Pi \Rightarrow \Sigma}
$$

where $(Inf2)$ is allowed only if below $(Inf1)$ is applied.
a is fresh in $(Inf1)$ so it is not present in Π, Σ (but it may appear in Γ, Δ).

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EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

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a is fresh in $(Inf1)$ so it is not present in Π, Σ (but it may appear in Γ, Δ).

Note that $(Inf2)$ is not allowed in separation since:

(a) either it is too weak for proving InfAx (if a is required fresh) or
(b) too strong – we cannot prove its derivability from InfAx (if a is arbitrary).

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

Proof of the Axiom of Infinity:

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

Proof of the Axiom of Infinity:

$$\dfrac{b \in a \Rightarrow b \in a \qquad \{b\} \in a \Rightarrow \{b\} \in a}{b \in a \Rightarrow \{b\} \in a}\ (Inf2)$$

$$\dfrac{b \in a \Rightarrow \{b\} \in a}{\Rightarrow b \in a \to \{b\} \in a}\ (\Rightarrow\to)$$

$$\dfrac{\Rightarrow b \in a \to \{b\} \in a}{\Rightarrow \forall x(x \in a \to \{x\} \in a)}\ (\Rightarrow \forall)$$

$$\dfrac{\emptyset \in a \Rightarrow \emptyset \in a \qquad \Rightarrow \forall x(x \in a \to \{x\} \in a)}{\emptyset \in a \Rightarrow \emptyset \in a \wedge \forall x(x \in a \to \{x\} \in a)}\ (\Rightarrow \wedge)$$

$$\dfrac{\emptyset \in a \Rightarrow \emptyset \in a \wedge \forall x(x \in a \to \{x\} \in a)}{\emptyset \in a \Rightarrow \exists y(\emptyset \in y \wedge \forall x(x \in y \to \{x\} \in y))}\ (\Rightarrow \exists)$$

$$\dfrac{\emptyset \in a \Rightarrow \exists y(\emptyset \in y \wedge \forall x(x \in y \to \{x\} \in y))}{\Rightarrow \exists y(\emptyset \in y \wedge \forall x(x \in y \to \{x\} \in y))}\ (Inf1)$$

Note that it does not work if $(Inf2)$ is a separate rule with a required fresh (case (a)).

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

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Derivability of $(Inf1) + (Inf2)$:

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

Derivability of $(Inf1) + (Inf2)$:

$$
\dfrac{\Rightarrow InfAx \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, b \in a \qquad \{b\} \in a, \Gamma \Rightarrow \Delta}{b \in a \to \{b\} \in a, \Gamma \Rightarrow \Delta}(\to\Rightarrow)}{\forall x(x \in a \to \{x\} \in a), \Gamma \Rightarrow \Delta}(\forall\Rightarrow)}{D'}}{\emptyset \in a, \forall x(x \in a \to \{x\} \in a), \Pi \Rightarrow \Sigma}}{\emptyset \in a \wedge \forall x(x \in a \to \{x\} \in a), \Pi \Rightarrow \Sigma}(\wedge\Rightarrow)}{\exists y(\emptyset \in y \wedge \forall x(x \in y \to \{x\} \in y)), \Pi \Rightarrow \Sigma}(\exists\Rightarrow)}{\Pi \Rightarrow \Sigma}(Cut)
$$

where D' is like D but with $\forall x(x \in a \to \{x\} \in a)$ added in each antecedent.

Note that derivability holds only if a is fresh in $(Inf1)$ oth we cannot prove derivability for $(Inf2)$ in separation (case (b)).

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

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The Axiom of Regularity:

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The Axiom of Regularity:

$$(Reg)\ \frac{\Gamma \Rightarrow \Delta, c \in a \qquad \Gamma \Rightarrow \Delta, c \in b}{\dfrac{\dfrac{\Gamma \Rightarrow \Delta}{\mathcal{D}}}{a \in b, \Pi \Rightarrow \Sigma}}$$

where (Reg) is allowed only if below a is not present in Π, Σ

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The Axiom of Regularity:

$$(Reg)\ \frac{\Gamma \Rightarrow \Delta, c \in a \qquad \Gamma \Rightarrow \Delta, c \in b}{\dfrac{\Gamma \Rightarrow \Delta}{\dfrac{\mathcal{D}}{a \in b, \Pi \Rightarrow \Sigma}}}$$

where (Reg) is allowed only if below a is not present in Π, Σ

$$\frac{c \in a \Rightarrow c \in a \qquad c \in b \Rightarrow c \in b}{c \in a, c \in b \Rightarrow}\ (Reg)$$

$$\frac{c \in a, c \in b \Rightarrow}{c \in a \Rightarrow c \notin b}\ (\Rightarrow \neg)$$

$$\frac{c \in a \Rightarrow c \notin b}{\Rightarrow c \in a \to c \notin b}\ (\Rightarrow \to)$$

$$\frac{\Rightarrow c \in a \to c \notin b}{\Rightarrow \forall z(z \in a \to z \notin b)}\ (\Rightarrow \forall)$$

$$\frac{a \in b \Rightarrow a \in b \qquad \Rightarrow \forall z(z \in a \to z \notin b)}{a \in b \Rightarrow a \in b \wedge \forall z(z \in a \to z \notin b)}\ (\Rightarrow \wedge)$$

$$\frac{a \in b \Rightarrow a \in b \wedge \forall z(z \in a \to z \notin b)}{a \in b \Rightarrow \exists y(y \in b \wedge \forall z(z \in y \to z \notin b))}\ (\Rightarrow \exists)$$

$$\frac{a \in b \Rightarrow \exists y(y \in b \wedge \forall z(z \in y \to z \notin b))}{\exists yy \in b \Rightarrow \exists y(y \in b \wedge \forall z(z \in y \to z \notin b))}\ (\exists \Rightarrow)$$

$$\frac{\exists yy \in b \Rightarrow \exists y(y \in b \wedge \forall z(z \in y \to z \notin b))}{\Rightarrow \exists yy \in b \to \exists y(y \in b \wedge \forall z(z \in y \to z \notin b))}\ (\Rightarrow \to)$$

$$\frac{\Rightarrow \exists yy \in b \to \exists y(y \in b \wedge \forall z(z \in y \to z \notin b))}{\Rightarrow \forall x(\exists yy \in x \to \exists y(y \in x \wedge \forall z(z \in y \to z \notin x)))}\ (\Rightarrow \forall)$$

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The Axiom Schema of Replacement:

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The Axiom Schema of Replacement:

φ is functional in t if for every $x \in t$ it assigns exactly one element. Formally:

$$FU(\varphi, t) \Leftrightarrow \forall x(x \in t \rightarrow \forall yz(\varphi(xy) \wedge \varphi(xz) \rightarrow y = z))$$

It is characterised by the pair of rules:

$$(\Rightarrow FU) \quad \frac{a_1 \in t, \varphi(a_1 a_2), \varphi(a_1 a_3), \Gamma \Rightarrow \Delta, a_2 = a_3}{\Gamma \Rightarrow \Delta, FU(\varphi, t)}$$

$(FU \Rightarrow)$

$$\frac{\Gamma \Rightarrow \Delta, b_1 \in t \quad \Gamma \Rightarrow \Delta, \varphi(b_1 b_2) \quad \Gamma \Rightarrow \Delta, \varphi(b_1 b_3) \quad b_2 = b_3, \Gamma \Rightarrow \Delta}{FU(\varphi, t), \Gamma \Rightarrow \Delta}$$

where a_1, a_2, a_3 are fresh parameters.

$$(Rep) \quad \frac{\Gamma \Rightarrow \Delta, FU(\varphi, b) \quad a = \{x : \exists y(y \in b \wedge \varphi(yx))\}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where a is fresh.

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The Axiom of Choice (in terms of the choice function):

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The Axiom of Choice (in terms of the choice function):

Informally: for every family x of nonempty sets there exists a function f, called choice function for x, such that for every $y \in x$, $f(y) \in y$.

Auxiliary notions of nonempty set and choice function:

$$N(t) \Leftrightarrow \forall y(y \in t \rightarrow \exists z(z \in y))$$

$$\forall xy(N(x) \rightarrow (y \in CF(x) \leftrightarrow y \in FUN(x, \bigcup x) \wedge \forall zz'(\langle z, z'\rangle \in y \rightarrow z' \in z)))$$

In terms of these defined notions we can formulate Choice Axiom as follows:

$$\forall x(N(x) \rightarrow \exists y(y \in CF(x)))$$

EXTENSIONS TO SERIOUS SET THEORY – THE CASE OF ZFC

The Axiom of Choice – respective rules:

The Axiom of Choice – respective rules:

$$(\Rightarrow N2)\ \cfrac{\Gamma \Rightarrow \Delta, b' \in a}{\cfrac{\Gamma \Rightarrow \Delta}{\mathcal{D}}}$$

$$(\Rightarrow N1)\ \frac{a \in t, \Pi \Rightarrow \Sigma, b \in a}{\Pi \Rightarrow \Sigma, N(t)}$$

$$(N \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, b \in t \qquad a \in b, \Gamma \Rightarrow \Delta}{N(t), \Gamma \Rightarrow \Delta} \qquad (CA)\ \frac{\Gamma \Rightarrow \Delta, N(t) \qquad a \in CF(t), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(CF \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, N(t) \qquad c \in FUN(t, \bigcup t), \forall xx'(\langle x, x' \rangle \rightarrow x' \in x), \Gamma \Rightarrow \Delta}{c \in CF(t), \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow CF)\ \frac{\Gamma \Rightarrow \Delta, N(t) \qquad \Gamma \Rightarrow \Delta, c \in FUN(t, \bigcup t) \qquad \Gamma \Rightarrow \Delta, \forall xx'(\langle x, x' \rangle \rightarrow x' \in x)}{\Gamma \Rightarrow \Delta, c \in CF(t)}$$

where a is fresh in $(CA), (N \Rightarrow), (\Rightarrow N1)$ but not in $(\Rightarrow N2)$. The latter rule can be applied only over the application of $(\Rightarrow N1)$ to which it is linked by having the same parameter a.

CONCLUDING REMARKS

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3. Formalisation of other variants of theory of classes (Tennant's system, Scott's variant of VTC) and their comparison.
4. Extending these approach to cover "serious theories of sets" like ZFC, NGB or NF.

Funded by the European Union (ERC, ExtenDD, project number: 101054714). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

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