Conservative Intuitionistic Epsilon Calculus over Lambda Abstraction

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ExtenDD Seminar
8 October 2025

Introduction

Epsilon Calculus was first developed for Hilbert's Program. There, ε-terms represent 'ideal' objects witnessing mathematical properties and embedding quantification.

Early works introduced procedures for eliminating ε-terms in derivations of ε-free theorems (1st correct proof of Herbrand's Theorem as a corollary) and to systematically substitute ε-terms for 'concrete' instances. A semantics based on arbitrary choice functions was later introduced.

Applications of Epsilon Calculus and its ε-terms involve (Avigad and Zach, 2020):

- foundations and philosophy of mathematics;
- philosophy of language and linguistics;
- philosophy of science;
- theorem provers;
- ...

Introduction

Formulas with occurrences ε-terms are strictly more expressive than first-order quantifiers. This is because the formulas in the scope of the ε operator are the extensions from which their witnesses, if any exists, are chosen from.

As a consequence, ε-terms can represent all Skolem functions of a language. In set theory, they imply the axiom of Choice in **ZF** and Global Choice in languages with of class variables and class abstracts (Bernays 1958).

It then comes as expected that extending Intuitionistic logic with ε-terms is not conservative over Intuitionistic quantification. There seem to be two reasons for this:

1. ε-terms may encode information that Intuitionistic logic is not able to handle;
2. ε-terms do not possess a proper scoping mechanism, and Intuitionistic logic is sensible to quantifier shifts.

Introduction

Several approaches were developed to approximate conservativity of ε-terms over Intuitionistic quantification:

- Restrictions of ε-terms formation or axioms (for a survey, see Mints, 2015);
- Conservativity for language extensions with an existence predicate (Mints, 2015);
- Conservativity over some stronger intermediate logics (Baaz and Zach, 2022).

Note that of these approaches provides ε-terms with a scoping mechanism, and that Mints' (2015) result does not extend to $=$.

Here, I develop a new approach using predicate abstraction to provide scopes for $\wedge$-terms. This approach was implicitly suggested in an early paper of Fitting (1975) on a conservative modal extension of Epsilon Calculus.

In this way, I obtain a conservative extension of Intuitionistic logic over ε-terms applied to λ-abstraction which supports $=$, thus opening to applications to theories.

Review of ε-Calculus

Take a language for (quantifier-free) Predicate logic, and extend it to *epsilon terms*:

$$\varepsilon x\, A \qquad (x \text{ is bound by } \varepsilon \text{ in } A)$$

Now, take an axiomatization of (quantifier-free) Classical logic in the extended language, and add the following axiom:

$$A(t/x) \to A(\varepsilon x\, A/x) \qquad \text{(Crit. Formula)}$$

The result $\boldsymbol{\varepsilon}\mathbf{C}$ is Hilbert's Epsilon Calculus. It conservatively extends both quantifier-free **C** and quantified **QC** Classical logic embedding quantifiers as follows:

$$\exists x\, A :\leftrightarrow A(\varepsilon x\, A/x) \qquad \forall x\, A :\leftrightarrow A(\varepsilon x \neg A/x)$$

Theorem

For any $A \in \mathcal{L}_{\mathbf{QC}}$*:* $\vDash_{\mathbf{QC}} A$ *iff* $\vDash_{\boldsymbol{\varepsilon}\mathbf{C}} A$

Failure of Intuitionistic Conservativity

Extend a (quantifier-free) Predicate language in a more careful way, so to include terms

$$\varepsilon x\, A \qquad\qquad \tau x\, A$$

Now, take an axiomatization **H** of (quantifier-free) Intuitionistic logic in the extended language, and add the following axioms:

$$A(t/x) \to A(\varepsilon x\, A/x) \qquad\qquad A(\tau x\, A/x) \to A(t/x)$$

Define quantifiers over ε- and τ-terms as follows:

$$\exists x\, A \;:\leftrightarrow\; A(\varepsilon x\, A/x) \qquad\qquad \forall x\, A \;:\leftrightarrow\; A(\tau x\, A/x)$$

Thanks to the addition of τ-terms, the difference between quantifiers seems better accounted for. The resulting logic $\boldsymbol{\varepsilon}$**H**, however, is <u>not conservative</u> over **QH** (Baaz and Zach, 2022).

In particular, the following are not valid in **QH** ($x \notin \text{free}(B)$):

- $\forall x\, (A(x) \vee B) \to (\forall x\, A(x) \vee B)$ (Constant Domain)
- $\forall x\, \neg\neg A(x) \to \neg\neg\forall x\, A(x)$ (Kuroda's Principle)
- $(B \to \exists x\, A(x)) \to \exists x\, (B \to A(x))$
- $(\forall x\, A(x) \to B) \to \exists x\, (A(x) \to B)$

Kuroda's Principle is proved in $\boldsymbol{\varepsilon\tau}$**H** under translation in one line:

$$\neg\neg A(\tau x\, \neg\neg A(x)) \to \neg\neg A(\tau x\, A(x))$$

The proof works because the ambiguity of scope of the τ-terms in the Critical formula cannot faithfully represent that of Intuitionistic quantifiers and their constraints.

Can a scoping mechanism for both ε- and τ-terms be imposed to solve this issue?

Predicate λ-Abstraction

Starting from the '70s, *predicate abstractions* were used to properly represent *de dicto* and *de re* modalities:

$$(\lambda x\, A)t \qquad (x \text{ is bound by } \lambda \text{ in } A)$$

Predicates defined over formulas in the scope of λ-binders provide scope of applications for terms, making such distinctions expressible:

$$\Box(\lambda x\, A)t \qquad\qquad (\lambda x\, \Box A)t$$

The increased expressibility given by predicate abstraction does not only involve modal extension. As shown by Scales (1969), it can also be used to define application of terms under certain conditions, such as the existence of a referent.

In the case of Free logic, this provides a logic which is both Positive and Negative. This device can also be used to embed under conditions that can make a logic non-classical only with respect to its term applications (LR 2025).

Predicate λ-Abstraction

There is also another way in which predicate abstraction have been used which involve expressivity issues which may not be apparent at first.

As an example, in his early papers, Fitting noticed that, even in Quantified Modal logics where *de dicto* and *de re* distinctions are not considered board, λ-abstraction is a useful language extension.

Fitting, e.g., showed how Herbrand's Theorem cannot be proved for certain Quantified Modal logics such as **QS4** (i.e., Quantified **S4** with the Converse Barcan Formula) without resorting to λ-abstraction applied to 'intensional' terms (Fitting, 1973). This is because in the construction of the Herbrand's disjunction, terms have to be interpreted under scopes which could not be expressible otherwise.

In order to provide a proof of Smullyan's Fundamental Theorem (a variant of Herbrand's), Fitting (1975) formulated a conservative extension of quantified **S4** by epsilon terms over predicate abstraction (here slightly modified).

Quantified S4

Let $\mathcal{L}_{\textbf{QS4}}$ be generated by the following grammar (no function symbols for simplicity):

$$A \quad ::= \quad P^n x_1 \dots x_n \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \forall x\, A \mid \exists x\, A$$

The axiomatization of **QS4** extends that of **QC** in $\mathcal{L}_{\textbf{QS4}}$ by the following:

K $\Box(A \to B) \to (\Box A \to \Box B)$

T $\Box A \to A$

4 $\Box A \to \Box\Box A$

Nec If $\vdash A$, then $\vdash \Box A$

Models $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ for **QS4** are based on reflexive and transitive Krikpe frames with increasing domains s.t. $w\mathcal{R}w' \Rightarrow \mathcal{D}(w) \subseteq \mathcal{D}(w')$ for any $w, w' \in \mathcal{W}$ (adjustments for interpreting free variables omitted here).

Let $\vDash_{\textbf{QS4}}$ be the consequence relation defined over all **QS4** models. Then:

Theorem

For any $A \in \mathcal{L}_{\textbf{QS4}}$: $\vdash_{\textbf{QS4}} A$ iff $\vDash_{\textbf{QS4}} A$

S4 ε-Calculus over λ-Abstraction

$\mathcal{L}_{\varepsilon\lambda\mathbf{S4}}$ extends $\mathcal{L}_{\mathbf{QS4}}$ with λ-abstractions to which ε- and τ-terms are applied:

$$A \;::=\; P^n x_1 \dots x_n \mid \cdots \mid \Box A \mid \forall x\, A \mid \exists x\, A \mid (\lambda x\, A)(\varepsilon x\, A) \mid (\lambda x\, A)(\tau x\, A)$$

The axiomatization of $\varepsilon\lambda\mathbf{S4}$ extends that of **QS4** in $\mathcal{L}_{\varepsilon\lambda\mathbf{S4}}$ by

- Axioms for ε- and τ-terms (simplified by the presence of quantifiers):

  $\exists$Emb $\exists x\, A \leftrightarrow (\lambda x\, A)(\varepsilon x\, A)$ $\forall$Emb $(\lambda x\, A)(\tau x\, A) \leftrightarrow \forall x\, A$

  $\varepsilon\lambda$Crit $(\lambda x\, A)t \to (\lambda x\, A)(\varepsilon x\, A)$ $\tau\lambda$Crit $(\lambda x\, A)(\tau x\, A) \to (\lambda x\, A)t$

- Axioms for λ-abstractions distribution over propositional connectives:

  λDist $(\lambda x\, A \to B)t \leftrightarrow ((\lambda x\, A)t \to (\lambda x\, B)t)$

- Axioms for λ-abstractions reorder, contraction and trivial bindings (omitted here).

Fitting's models $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{F} \rangle$ for $\boldsymbol{\varepsilon\lambda}$**S4** extend those for **QS4** by a set $\mathcal{F}$ of functions f interpreting ε- and τ-terms s.t., for $\overline{d}$ being a name for a $d \in \mathcal{D}$:

- $f_{\varepsilon x\, A}(w) \in \mathcal{D}(w)$ and $(\exists d \in \mathcal{D}(w) : \mathcal{M}, w \Vdash A(\overline{d}/x)) \Rightarrow \mathcal{M}, w \Vdash A(\overline{f_{\varepsilon x\, A}(w)}/x)$
- $f_{\tau x\, A}(w) \in \mathcal{D}(w)$ and $\mathcal{M}, w \Vdash A(\overline{f_{\tau x\, A}(w)}/x) \Rightarrow (\forall d \in \mathcal{D}(w) : \mathcal{M}, w \Vdash A(\overline{d}/x))$

Hence, ε- and τ-terms applied to λ-abstractions are interpreted as follows:

- $\mathcal{M}, w \Vdash (\lambda x\, A)(\varepsilon x\, B)$ iff $\mathcal{M}, w \Vdash A(\overline{f_{\varepsilon x\, B}(w)})$
- $\mathcal{M}, w \Vdash (\lambda x\, A)(\tau x\, B)$ iff $\mathcal{M}, w \Vdash A(\overline{f_{\tau x\, B}(w)})$

Let $\vDash_{\boldsymbol{\varepsilon\lambda}\mathbf{S4}}$ be the induced consequence relation defined over all $\boldsymbol{\varepsilon\lambda}$**S4** models. Then:

Theorem (Fitting, 1975)

For any $A \in \mathcal{L}_{\boldsymbol{\varepsilon\lambda}\mathbf{S4}}$*:* $\vdash_{\boldsymbol{\varepsilon\lambda}\mathbf{S4}} A$ *iff* $\vDash_{\boldsymbol{\varepsilon\lambda}\mathbf{S4}} A$

Moreover, $\boldsymbol{\varepsilon\lambda}$**S4** is a conservative extension of **QS4** given how it extends its models:

Theorem (Fitting, 1975)

For any $A \in \mathcal{L}_{\mathbf{QS4}}$*:* $\vDash_{\mathbf{QS4}} A$ *iff* $\vDash_{\boldsymbol{\varepsilon\lambda}\mathbf{S4}} A$

The S4 Embedding

Recall the modal embedding of Intuitionistic logic **QH** in **QS4**:

$$
\begin{array}{rcl}
\mathcal{L}_{\mathbf{QH}} & & \mathcal{L}_{\mathbf{QS4}} \\
(P)^{\Box} & := & \Box P \\
(\neg A)^{\Box} & := & \Box\neg A^{\Box} \\
(A \wedge B)^{\Box} & := & (A^{\Box} \wedge B^{\Box}) \\
(A \vee B)^{\Box} & := & (A^{\Box} \vee B^{\Box}) \\
(A \to B)^{\Box} & := & \Box(A^{\Box} \to B^{\Box}) \\
(\exists x\, A)^{\Box} & := & \exists x\, A^{\Box} \\
(\forall x\, A)^{\Box} & := & \Box\forall x\, A^{\Box}
\end{array}
$$

Theorem

For any $A \in \mathcal{L}_{\mathbf{QH}}$: $\vdash_{\mathbf{QH}} A$ iff $\vdash_{\mathbf{QS4}} A^{\Box}$

Extending the S4 Embedding

By conservativity over **QS4**, the embedding immediately extends to the case of $\boldsymbol{\varepsilon\lambda}$**S4**:

$$
\begin{array}{rcl}
\mathcal{L}_{\mathbf{QH}} & & \mathcal{L}_{\boldsymbol{\varepsilon\lambda}\mathbf{S4}} \\
(P)^{\Box} & := & \Box P \\
 & \vdots & \\
(\exists x\, A)^{\Box} & := & (\lambda x\, A^{\Box})(\varepsilon x\, A^{\Box}) \\
(\forall x\, A)^{\Box} & := & \Box(\lambda x\, A^{\Box})(\tau x\, A^{\Box})
\end{array}
$$

Theorem (essentially Corollary of Fitting, 1975)

For any $A \in \mathcal{L}_{\mathbf{QH}}$*:* $\vdash_{\mathbf{QH}} A$ *iff* $\vdash_{\boldsymbol{\varepsilon\lambda}\mathbf{S4}} A^{\Box}$

Conservativity of **QH** over ε- and τ-terms is thus obtained through the embedding.

Conservative Intuitionistic ε-Calculus over λ-Abstraction

The previous results shows that a conservative embedding of Intuitionistic quantification with ε- and τ-terms can be obtained by extending the language of Epsilon Calculus with **S4**-modalities and λ-abstraction.

The syntax of $\mathcal{L}_{\boldsymbol{\varepsilon\lambda}\mathbf{S4}}$ and the previous modal translation, however, suggest a way to 'absorb' modalities into λ-abstraction.

In this way, and applying a term t to a new λ-abstractors $[\lambda x\, A]$ is interpretable as 'applying t after its construction'.

Since an ε-term may not be constructable at a point, λ-abstraction isolates the 'indefinite' part of the logic away. Given the scoping mechanism of predicate abstraction, no ambiguity arises on to against which exact constructions epsilon terms will be evaluated.

Conservative Intuitionistic ε-Calculus over λ-Abstraction

$\mathcal{L}_{\boldsymbol{\varepsilon\lambda}\mathbf{H}}$ extends $\mathcal{L}_{\mathbf{QH}}$ with λ-abstractions to which ε- and τ-terms are applied:

$$A \quad ::= \quad P^n x_1 \dots x_n \mid \cdots \mid \forall x\, A \mid \exists x\, A \mid [\lambda x\, A](\varepsilon x\, A) \mid [\lambda x\, A](\tau x\, A)$$

The axiomatization of $\boldsymbol{\varepsilon\lambda}\mathbf{H}$ extends that of **QH** in $\mathcal{L}_{\boldsymbol{\varepsilon\lambda}\mathbf{H}}$ by:

- Axioms for ε- and τ-terms (simplified by the presence of quantifiers):

  $\exists$Emb $\quad \exists x\, A \leftrightarrow [\lambda x\, A](\varepsilon x\, A)$ $\qquad$ $\forall$Emb $\quad [\lambda x\, A](\tau x\, A) \leftrightarrow \forall x\, A$

  $\varepsilon\lambda$Crit $\quad [\lambda x\, A]t \rightarrow [\lambda x\, A](\varepsilon x\, A)$ $\qquad$ $\tau\lambda$Crit $\quad [\lambda x\, A](\tau x\, A) \rightarrow [\lambda x\, A]t$

- Axioms for λ-abstractions distribution:

  $\Box\lambda$Dist $\quad [\lambda x\, A \rightarrow B]t \rightarrow ([\lambda x\, A]t \rightarrow [\lambda x\, B]t)$

- Axioms for λ-abstractions reorder, contraction and trivial bindings (omitted here).

Models $\mathcal{M} = \langle \mathcal{W}, \leq, \mathcal{D}, \mathcal{I}, \mathcal{F} \rangle$ for $\boldsymbol{\varepsilon\lambda}\mathbf{H}$ extend those for **QH** by a set $\mathcal{F}$ of functions f interpreting ε- and τ-terms s.t., for $\overline{d}$ being a name for a $d \in \mathcal{D}(w)$:

- $f_{\varepsilon x\, A}(w) \in \mathcal{D}(w)$ and $(\exists d \in \mathcal{D}(w) : \mathcal{M}, w \Vdash A(\overline{d}/x)) \Rightarrow \mathcal{M}, w \Vdash A(\overline{f_{\varepsilon x\, A}(w)}/x)$
- $f_{\tau x\, A}(w) \in \mathcal{D}(w)$ and $\mathcal{M}, w \Vdash A(\overline{f_{\tau x\, A}(w)}/x) \Rightarrow (\forall d \in \mathcal{D}(w) : \mathcal{M}, w \Vdash A(\overline{d}/x))$

Hence, ε- and τ-terms applied to λ-abstractions are interpreted as follows:

- $\mathcal{M}, w \Vdash [\lambda x\, A](\varepsilon x\, B)$ iff $\mathcal{M}, w \Vdash A(\overline{f_{\varepsilon x\, B}(w)})$
- $\mathcal{M}, w \Vdash [\lambda x\, A](\tau x\, B)$ iff $\forall w' : w \leq w' \Rightarrow \mathcal{M}, w' \Vdash A(\overline{f_{\tau x\, B}(w')})$

Let $\vDash_{\boldsymbol{\varepsilon\lambda}\mathbf{H}}$ be the induced consequence relation defined over all $\boldsymbol{\varepsilon\lambda}\mathbf{H}$ models. Then:

Theorem

For any $A \in \mathcal{L}_{\boldsymbol{\varepsilon\lambda}\mathbf{H}}$: $\vdash_{\boldsymbol{\varepsilon\lambda}\mathbf{H}} A$ iff $\vDash_{\boldsymbol{\varepsilon\lambda}\mathbf{H}} A$

Finally, $\boldsymbol{\varepsilon\lambda}\mathbf{H}$ is a conservative extension of **QH** given how it extends its models:

Theorem

For any $A \in \mathcal{L}_{\mathbf{QH}}$: $\vDash_{\mathbf{QH}} A$ iff $\vDash_{\boldsymbol{\varepsilon\lambda}\mathbf{H}} A$

Representation of Invalid Quantifier Shifts

$$\forall x\, (A(x) \vee B) \to (\forall x\, A(x) \vee B) \qquad (\textbf{QH})$$
$$\Box\forall x\, (A(x) \vee B) \to (\Box\forall x\, A(x) \vee B) \qquad (\textbf{QS4})$$
$$\Box(\lambda x\, A(x) \vee B)(\tau x\, (A(x) \vee B)) \to (\Box(\lambda x\, A(x))(\tau x\, A(x)) \vee B) \qquad (\boldsymbol{\varepsilon\lambda}\textbf{QS4})$$
$$[\lambda x\, A(x) \vee B](\tau x\, (A(x) \vee B)) \to ([\lambda x\, A(x)](\tau x\, A(x)) \vee B) \qquad (\boldsymbol{\varepsilon\lambda}\textbf{H})$$

$$\forall x\, \neg\neg A(x) \to \neg\neg\forall x\, A(x) \qquad (\textbf{QH})$$
$$\Box\forall x\, \Box\neg\Box\neg A(x) \to \Box\neg\Box\neg\Box\forall x\, A(x) \qquad (\textbf{QS4})$$
$$\Box(\lambda x\, \Box\neg\Box\neg A(x))(\tau x\, \Box\neg\Box\neg A(x)) \to \Box\neg\Box\neg\Box(\lambda x\, A(x))(\tau x\, A(x)) \qquad (\boldsymbol{\varepsilon\lambda}\textbf{QS4})$$
$$[\lambda x\, \neg\neg A(x)](\tau x\, \neg\neg A(x)) \to \neg\neg[\lambda x\, A(x)](\tau x\, A(x)) \qquad (\boldsymbol{\varepsilon\lambda}\textbf{H})$$

$$(\forall x\, A(x) \to B) \to \exists x\, (A(x) \to B) \qquad (\textbf{QH})$$
$$(\Box\forall x\, A(x) \to B) \to \exists x\, \Box(A(x) \to B) \qquad (\textbf{QS4})$$
$$(\Box(\lambda x\, A(x))(\tau x\, A(x)) \to B) \to (\lambda x\, \Box(A(x) \to B))(\varepsilon x\, \Box(A(x) \to B)) \qquad (\boldsymbol{\varepsilon\lambda}\textbf{QS4})$$
$$([\lambda x\, A(x)](\tau x\, A(x)) \to B) \to [\lambda x\, A(x) \to B](\varepsilon x\, A(x) \to B) \qquad (\boldsymbol{\varepsilon\lambda}\textbf{H})$$

x not free in B

Conclusions

The simple extension of Intuitionistic logic by epsilon terms is non-conservative, as intuitionistically invalid quantifier shifts become provable.

Various approaches have been proposed to recover conservativity, but none satisfactory for developing theories. I proposed a new one implicit in a early work of Fitting conservatively extending **S4** to epsilon terms over predicate abstraction.

While overlooked by Fitting, this provides a conservative extension of quantified Intuitionistic logic over the modal embedding. I expanded Fitting's work by absorbing **S4** modalities into a new kind of predicate abstraction which provides the right scoping mechanism for epsilon terms.

The result is a conservative Intuitionistic logic with epsilon terms over predicate abstraction which preserves conservativity even when extended by identity and extensionality.

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