Descriptions in Informational Semantics for Relevant Logic

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The Paradoxes of Implication

Relevant Logic was constructed to avoid the paradoxes of implication:

- $A \to (B \to A)$ (positive paradox)
- $\neg A \to (A \to B)$ (negative paradox)
- $(A \wedge \neg A) \to B$ (explosion)
- $A \to (B \to B)$

The Ternary Accessibility Relation

If we adopt a binary accessibility to characterise implication, we naturally end up with some of the paradoxes: If $a \vDash A$ then it still will be the case that $a \vDash B \to B$ for arbitrary A and B.

So, Routley and Meyer employed a ternary relation, R, on situations:

$$a \vDash A \to B \text{ iff} \forall x \forall y ((Raxy \wedge x \vDash A) \Rightarrow y \vDash B)$$

Incompatibility and The Routley Star

Situations as being compatible or incompatible with one another. Two incompatible situations cannot occur at the same possible world.

We add an incompatibility relation $\perp$ on situations, but it is easier (formally) to add the Routley-star operator: A situation a^* is the *maximal situation compatible with a*.

$$a \vDash \neg A \text{ iff } a^* \nvDash A$$

Partiality and Inconsistency

In order for the semantics to avoid the paradoxes, for any formula A there must be models in which there are situations that do not satisfy A. And, in order to avoid ex falso, there must be situations in which contradictions are satisfied but some other formulae are not satisfied.

The Routley-Meyer Semantics allows situations that are partial and situations that are inconsistent (some can be both).

But, it also needs to somehow validate the theorems of the logics. So, it distinguishes between normal and non-normal situations. A formula is valid on a class of models iff it is satisfied by every normal situations in those models.

There are two sorts of partiality that are important for the present topic:

1. Failure of Bivalence
2. Non-Normality (not all valid formulae are in every situation)

Normal Situations

R-M define a hereditariness relation $\leq$, such that $a \leq b$ iff there is some normal situation n, $Rnab$.

They add conditions to satisfy a general hereditariness condition: For all formulae A and all situations a and b, if $a \vDash A$ and $a \leq b$, then $b \vDash A$.

So, in all normal worlds, $A \to A$ hold, but also $A \to B$ for all general closure conditions on situations.

Routley-Meyer Semantics

A model is a tuple $\langle S, N, R, *, I\rangle$, where S is a non-empty set (the set of situations), N is a non-empty subset of S (the set of normal situations), $R \subseteq S^2$, $*$ is a unary operator on S, and I is an interpretation.

$$a \leq b \text{ iff } \exists x(x \in N \wedge Rxab)$$

Conditions on Frames:

1. If $Rabc$ and $a' \leq a$ then $Ra'bc$;
2. If $a \leq b$ then $b^* \leq a^*$;
3. $a^{**} = a$;
4. $a \in I(p)$ and $a \leq b$ then $b \in I(p)$.

Satisfaction Conditions

- $a \models p$ if and only if $a \in I(p)$
- $a \models A \wedge B$ if and only if $a \models A$ and $a \models B$
- $a \models \neg A$ if and only if $a^* \not\models A$
- $a \models A \rightarrow B$ if and only if, for all $b \models A$, if $Rabc$ then $c \models B$

A formula A is valid on a model in and only if $N \subseteq [\![A]\!]_I$.

Bivalence

If we want to make valid the law of excluded middle, we add

$$\forall x(x \in N \Rightarrow x^* \leq x)$$

But we might not want that for a theory of definite descriptions.

Information and Partiality

I adapt Kripke's interpretation of his semantics for intuitionist logic. He takes the "worlds" in his semantics to be *evidential situations*. I treat the "worlds" (points, setups) in the Routley-Meyer semantics as situations as well, but drop the notion of evidence in favour of the information available in an environment. (When I first adopted the informational interpretation, I used Barwise and Perry's situation semantics, but not very much of their relational theory of meaning, etc., has survived in the present theory.)

Situations hold or do not hold at ((im?)possible) worlds. They capture some or all of the information available in the world. Assumption 1: every fact counts as information. Assumption 2: information is always true.

Information and Implication

$A \rightarrow B$ is read as saying that

- A carries the information that B
- A licenses the inference to B.

The ternary relation: Let Rab be the set of situations such that $Rabc$. The information in a, when applied to b, tells us that if a and b are both true in the same world, then at least one of the situations in $Rabc$ is true in that world as well.

Normality

A normal situation is one that contains (accurate) information about the whole frame. The information that it contains is the information about what constitutes a situation for the frame. It contains the general closure conditions on situations.

Information Conditions versus Truth Conditions

The semantics is supposed to be understood in terms of the objective or environmental information available in a situation. The idea of objective information is that it need not be just an entry in a database, but rather an available fact. But there is a difference between what constitutes information and what makes a sentence true.

Here is a Russellian argument for this claim:

Example: All the dogs from the neighbourhood are in the park.

Suppose that there are only two dogs in the neighbourhood: Nova and Milo. 'Nova is in the park and Milo is in the park' does not by itself entail that every dog in the neighbourhood is in the park. We need the general information that Nova and Milo are the only dogs in the neighbourhood.

The Semantics of Quantification

In the 1970s, Routley attempted to prove completeness for quantified relevant logics over R-M frames with standard constant domain semantics, but in the early 1980s, Kit Fine proved that these logics are in fact incomplete over this semantics. Fine produced an alternative semantics, and later Rob Goldblatt and I produced an alternative semantics.

We add to the definition of a frame a domain of individuals, D, a set of propositions ($Prop$), and a set of propositional functions. A proposition is a set of situations closed upwards under $\leq$, but typically not all upsets are included in $Prop$.

Information Condition for the Universal Quantifier

$a \vDash \forall x A(x)$ if and only if there is some $\pi \in \mathit{Prop}$ such that (i) $\pi \subseteq \bigcap_{i \in D} [\![A(x)]\!]_{f[i/x]}$ and (ii) $a \in \pi$.

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This condition says that $\forall x A(x)$ is satisfied by a iff there is some proposition at a that entails every instance of $A(x)$.

The Existential Quantifier

We adopt the usual definition of the existential quantifier:

$$\exists x A \ =_{df} \ \neg\forall x \neg A$$

The derived information condition for $\exists$ is:

$a \vDash \exists x A$ iff for all $\pi \in Prop$, if $\bigcup_{i \in D} [\![A(x)]\!]_{f[i/x]} \subseteq \pi$ then $a \in \pi$.

This says that $\exists x A$ holds in a if and only if every proposition that is entailed by every instance of $A(x)$ obtains in a.

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But note that we can constrain normal situations such that they do always contain witnesses for the existential statements (this works only for some logics).

Theories of Definite Descriptions

1. Definite Descriptions as Incomplete Symbols.
2. Definite Descriptions as Singular Terms.

A Russellian Definition

Here is a relevant version of PM $*14.01$:

$$[(\imath x)\varphi(x)]\psi(\imath x)\varphi(x) \ =_{Df} \ \exists x(\varphi(x) \wedge \forall y(\varphi(y) \mapsto y = x) \wedge \psi(x))$$

What is $\mapsto$?

Should $\mapsto$ just be $\rightarrow$? What '$\forall y(\varphi(y) \rightarrow y = x)$' says is that $\varphi(y)$ carries the information that y is identical to x for every y. That seems rather strong.

We might want to add a second weaker implication connective to the logic by adding a second ternary relation.

Descriptions and Reference

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But having a statement like 'The φ is ψ' satisfied in a situation *a* does presuppose that there are situations extending *a* in which there is really a unique φ.

Information Who?

We might know that someone gave the proof that constitutes the Fitch Knowability Paradox, but not know who it was. We can use the description 'the prover of the Fitch paradox' without knowing who this prover was.

A situation that contains information about the prover of the Fitch paradox but does not attribute to anyone in particular this proof can be thought of as being a state one occupies in which one has much of the information about this prover, except does not know who he or she is (it was Alonzo Church).

Descriptions as Singular Terms

Because the semantics includes non-bivalent situations, it may be that it is compatible with an approach that treats descriptions as singular terms and will allow for semantic presupposition failure. But we still have the problem of unwitnessed existentials, so we can have a problem making the following valid:

$$\exists x(\psi(x) \wedge \forall y(\psi(y) \mapsto y = x) \wedge \varphi(x))) \rightarrow \varphi(((\imath x)\psi(x)).$$

Towards a Semantics for Definite Descriptions

We *might* be able to constrain descriptions to work normally at normal situations. To do so we can add a constant, t ("Ackermann t") that holds at all and only normal situations. We then add t into the axioms at strategic locations, such as:

$$(\exists x(\psi(x) \wedge \forall y(\psi(y) \mapsto y = x) \wedge \varphi(x))) \wedge t) \to \varphi(((\imath x)\psi(x)).$$

What happens to definite descriptions at Non-Normal Worlds?

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I don't know.

The Semantics of Identity: The Problem

Adopting the standard truth condition for identity as an information condition has irrelevant consequences. Suppose that $a \vDash i = j$ if and only if $I_a(i) = I_a(j)$. Then, we make valid

$$A \rightarrow i = i$$

$$\neg i = i \rightarrow A$$

for arbitrary formulas A.

Strategies for the Semantics of Identity

1. We could treat identity as a relation like any other, but with some constraints, such as symmetry and some form of transitivity. Substitution has to be forced, but it is doable (see Standefer (2021) Mares (1992)).
2. Adopt a "Fregean" theory of identity, according to which identity information is about coincidence of intensional entities.

Intensional Objects

Let f be a partial function from situations and individual constants to individuals in the domain. Where c is an individual constant, $f(c)$ is a *intensional object* – a partial function from situations to sets of individuals.

$$a \models P(c) \text{ iff } \exists x \in f_a(c) \text{ and } c \in I_a(P)$$

$$a \models c = d \text{ iff } f_a(c) = f_a(d)$$

We now can falsify

$$A \to c = c$$

and

$$\neg c = c \to A$$

for some formulas A.

But ...

We do have a minor problem with substitution and negation: We get

$$(A(c) \wedge c = d) \rightarrow A(d)$$

when A does not contain implication, negation, or quantifiers. We don't want it implicational formulas, but we do (?) want it for negation. So, we have to set

$$f_a(c) = f_{a^*}(c)$$

where $f_a(c)$ exists.

Identity and Quantification

We do want

$$(\forall x P(x,c) \wedge c = d) \rightarrow \forall x P(x,d)$$

E.g.,

$$(\forall x(0 \leq x) \wedge 0 = 6 - (3+3)) \rightarrow \forall x(6 - (3+3) \leq x).$$

One way to make this valid is to alter the information clause for the quantifier to restrict the situations what we look at to determine that there is some proposition π that entails all instances of $P(x,c)$. When we are determining whether a contains the information that $\forall x P(x,c)$ we could look only at what occurs in situations b in which all the identities that obtain in a also obtain in b. This *seems* to work, but I haven't been playing with it for very long!

A Problem Concerning Indefinite Descriptions

Hilbert's Axiom:

$$A(i) \rightarrow A(\epsilon x A(x))$$

Assumption: A formula A is such that $A(k) \rightarrow \neg A(k)$ for all terms k.

1.	$A(i) \rightarrow A(\epsilon x A(x))$	Axiom
2.	$\neg A(\epsilon x A(x)) \rightarrow \neg A(j)$	Axiom contraposed
3.	$A(\epsilon x A(x)) \rightarrow \neg A(\epsilon x A(x))$	Assumption
4.	$A(i) \rightarrow A(j)$	1,2,3

Thank you / Danke / Dziękuję