Some Thoughts on Formalising Definite Descriptions with Binary Quantifiers in Modal Logic

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Definite Descriptions formalised by Binary Quantification

Sentences containing definite descriptions can be formalised by a binary quantifier that binds a variable and forms a formula from two formulas:

(1) The F is G
(2) $Ix[F, G]$

This has the advantage of building scope distinctions immediately into the notation:

(3) External negation: $\neg Ix[F, G]$
(4) Internal negation: $Ix[F, \neg G]$

(5) Modality *de dicto*: $\Box Ix[F, G]$: It is necessary that the F is G
(6) Modality *de re*: $Ix[F, \Box G]$: The F is necessarily G

Descriptions formalised by Term Forming Operators. I

Definite descriptions are commonly formalised by the $\imath$ operator. It binds a variable and forms a singular term from a formula:

(1) The F
(2) $\imath xF$

Sentences containing definite descriptions are then formalised by attaching $\imath$ terms to predicate letters:

(3) The F is G
(4) $G\imath xF$

This is how the $\imath$ operator was introduced by Peano and how definite descriptions are now mostly formalised, in particular in formalisations of theories of definite descriptions in free logic by Hintikka, Lambert, van Fraassen and others. Russell used the $\imath$ operator, too, but with a caveat.

Descriptions formalised by Term Forming Operators. II

The caveat concerning Russell is that he only introduced the $\imath$ operator after having explained the meaning of definite descriptions in the context of complete sentences in which they occur, and this explanation required markers for scope distinctions.

Some free logicians abhor scope distinctions, but it is clear that they are needed in quantified modal logic with definite descriptions. To mark scope, it is necessary to introduce a further device into the language. Often the λ operator is used for this purpose. Formulas with λ abstracts are formed as follows: if A is a formula, $\lambda x A$ is a λ abstract, and if t is a term, $\lambda x A(t)$ is a formula.

(1) Modality *de dicto*: $\Box \lambda x G(\imath x F)$
(2) Modality *de re*: $\lambda x \Box G(\imath x F)$

Plan of the Talk

First, I'll present a prominent system of quantified modal logic with definite descriptions in which they are formalised as usual by the $\imath$ operator and scope is marked by λ: this is the system presented by Garson in his *Modal Logic for Philosophers*.

Then, I'll present how modal logic can be extended by a binary quantifier for formalising complete sentences in which definite descriptions occur.

Next, I'll compare the two approaches, in particular with respect to the important notion of rigidity.

I'll end with a few words comparing my system to Fitting and Mendelsohn's system of *First Order Modal Logic* (second edition).

Garson's System. I

The rules for the connectives $\neg, \rightarrow$ are those for classical logic.
The rules for propositional modal logic **K** are standard:

$$(\rightarrow I)\ \dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \rightarrow B}\,i \qquad (\rightarrow E)\ \dfrac{A \rightarrow B \qquad A}{B} \qquad (\bot E^C)\ \dfrac{\begin{matrix}[\neg A]^i \\ \Pi \\ \bot\end{matrix}}{A}\,i$$

$$(\Box I)\ \dfrac{\Box A_1 \ldots \Box A_n \qquad \begin{matrix}\underbrace{[A_1]^{i_1} \ldots [A_n]^{i_n}} \\ \Pi \\ B\end{matrix}}{\Box B}\,{}_{i_1,\ldots i_n} \qquad \text{where } n \geq 0$$

T, **S4**, **B**, **S5**: add the usual axioms/rules/modifications of rules

Garson's System. II

The rules for the quantifiers and identity adapt those of standard positive free logic **PFL**. An *atomic term* is a constant or parameter. A *predication* is a predicate letter or λ abstract.

$$(\forall I) \quad \dfrac{\begin{array}{c}[\exists ! a]^i \\ \Pi \\ A^x_a\end{array}}{\forall x A}\, i \qquad\qquad (\forall E) \quad \dfrac{\forall x A \qquad \exists ! c}{A^x_c}$$

Restrictions on $\forall I$ are as usual. In $\forall E$, c is an atomic term.

$$(= I) \quad \dfrac{}{t = t} \qquad\qquad (= E) \quad \dfrac{s = t \qquad A^x_s}{A^x_t}$$

where A is a predication and t, s are any terms.

The restrictions on $(\forall E)$ and $(= E)$ are there to give rules sensible in quantified modal logic: consider the case where x is in the scope of a modal operator in A and the instantiating terms are not rigid.

Garson's System. III

Constants and parameters are interpreted rigidly:

$$(RC_1) \quad \frac{c = d}{\Box\, c = d} \qquad\qquad (RC_2) \quad \frac{\neg\, c = d}{\Box\neg\, c = d}$$

where c, d are atomic terms.

We can always refer to the referent of a term rigidly by picking a fresh parameter and assuming it to have the same referent:

$$\begin{array}{c} [a = t]^i \\ \Pi \\ (\exists i) \quad \dfrac{A}{A}\, i \end{array}$$

where a is a parameter that does not occur free in A, t nor any undischarged assumptions of Π except $a = t$.

Garson's System. IV

The rules for λ operator are almost those one would expect:

$$(\lambda I)\ \frac{A^x_c}{\lambda x A(c)} \qquad\qquad (\lambda E)\ \frac{\lambda x A(c)}{A^x_c}$$

where c is an atomic term.

Hence $\vdash \lambda x \Box A(c) \leftrightarrow \Box A^x_c$ and $\vdash \Box A^x_c \leftrightarrow \Box \lambda x A(c)$.

The restriction is needed because modalities *de re* and *de dicto* are not equivalent for definite descriptions:

(1) $\Box \lambda x A(\imath x F)$: It is necessary that being A is true of the F.
(2) $\lambda x \Box A(\imath x F)$: Being necessarily A is true of the F.

There are no function symbols, so the only non-rigid terms are definite descriptions. (Function symbols are not needed: functions can be defined in terms of a relation and a definite description.)

Garson's System. V

The $\imath$ operator is governed by rules standard for positive free logic:

$$(\imath I)\quad \dfrac{A^x_c \qquad \exists! c \qquad \begin{array}{c}[A^x_a]^i \; [\exists! a]^j \\ \Pi \\ c = a\end{array}}{c = \imath x A}\, i,j$$

$$(\imath E_1)\quad \dfrac{c = \imath x A \qquad \exists! c}{A^x_c}$$

$$(\imath E_2)\quad \dfrac{c = \imath x A \qquad A^x_d \qquad \exists! c \qquad \exists! d}{c = d}$$

where c, d are atomic terms, and in $(\imath I)$, a is a parameter that does not occur in any undischarged assumption of Π except A^x_a and $\exists! a$.

The restriction to atomic terms corresponds to the restriction on $(\forall E)$ if we derive these rules from Lambert's Axiom:

$$\forall y (y = \imath x A \leftrightarrow (\forall x (A \leftrightarrow y = x)))$$

Garson's System. VI

The semantics for Garson's system is the standard semantics for variable domain quantified modal logic, so I won't go into the details: a structure $\mathcal{M}$ consists of a set $\mathcal{D}$ of objects, a set of worlds w, each with a domain $\mathcal{D}(w) \subseteq \mathcal{D}$ of objects that exist at that world, and an accessibility relation on them. The language is interpreted as usual: constants and predicate letters are assigned objects and subsets of $\mathcal{D}$, same for assignment functions for variables. I'll only give the clauses for $\imath$ and λ terms:

$d \in I^v(w, \lambda x A)$ iff for the x-variant v' of v s.th. $v'(x) = d$: $\mathcal{M}, w \Vdash_{v'} A$

$I^v(w, \imath x A) = d$, if there is a unique object d in $\mathcal{D}(w)$ such that for the x-variant v' of v that assigns d to x, $\mathcal{M}, w \Vdash_{v'} A$;
$I^v(w, \imath x A) \notin \mathcal{D}(w)$ otherwise.

A Comment on the Clause for $\imath$

$I^v(w, \imath xA) = d$, if there is a unique object d in $\mathcal{D}(w)$ such that for the x-variant v' of v that assigns d to x, $\mathcal{M}, w \Vdash_{v'} A$;
$I^v(w, \imath xA) \notin \mathcal{D}(w)$ otherwise.

If a unique F exists at a world w, then $\imath xA$ refers to it. If not, $\imath xA$ refers to an unspecified object in the outer domain of w.

As in Garson's system all terms refer, he has no other choice: there may not be an A in the outer domain of w.

(1) The round square is round.
(2) The round square is not round.

Which one is true at a world w depends on the random object to which the description 'the round square' refers.

This is an oddity of the system.

Comment on $(\exists i)$

Garson's $(\exists i)$ is admissible, but not derivable, in (non-modal) **PFL**.
(a) Any application of $(\exists i)$ can be transformed into a correct proof of **CPF** by replacing a by t and leaving out the rule:

$$
\begin{array}{c}
[a = t]^i \\
\Pi \\
(\exists i)\quad \dfrac{A}{A}\, i
\end{array}
\qquad \rightsquigarrow \qquad
\begin{array}{c}
t = t \\
\Pi^a_t \\
A
\end{array}
$$

The transformation does not lead to any violations of restrictions on rules, and closes all assumptions $t = t$ by $(= I)$.

(b) Consider a semantics for **PFL** that is like the standard one (outer domain $\mathfrak{D}$ and inner domain $\mathfrak{E} \subseteq \mathfrak{D}$ over which $\exists!$ is interpreted), but with an element $*$ not in the outer domain. Interpretation of the language stays untouched, i.e. all constants and $\imath$ terms are interpreted in the outer domain. Then all rules of **CPF** preserve validity. But $a = t$ is false, if $v(a) = *$, for any t, and so $(\exists i)$ doesn't preserve validity.

Formalising Definite Descriptions with a Binary Quantifier

Formalising definite descriptions with a binary quantifier takes on Russell's point that we shouldn't look for a referent of 'the F' first and then ask for the truth conditions of 'The F is G', but should start with the latter, i.e. with a complete sentence.

The present approach is un-Russellian in two respects:

(a) It is modal.
(b) It uses positive free logic.

Nonetheless, it takes on Russell's point about complete sentences.

Proposal: The minimal requirement we should impose on the truth of $Ix[F, G]$ is that there be a unique F. But it need not exist.

Thus 'The author of *Principia* smokes a pipe', 'The round square is round' and 'The round square is not round' are all false.

Natural Deduction Rules for I for Positive Free Logic

Where c, c_1, c_2 are atomic terms:

$$(II) \quad \dfrac{F^x_c \qquad G^x_c \qquad \begin{matrix} [F^x_a]^i \\ \Pi \\ a = c \end{matrix}}{Ix[F, G]}\ {\scriptstyle i}$$

where a is different from c and does not occur in F, G or any undischarged assumptions of Π except F^x_a.

$$(IE^1) \quad \dfrac{Ix[F, G] \qquad \begin{matrix} \underbrace{[F^x_a]^i,\ [G^x_a]^j} \\ \Pi \\ C \end{matrix}}{C}\ {\scriptstyle i,j}$$

where a is not free in C or any undischarged assumptions in Π except F^x_a, G^x_a.

$$(IE^2) \quad \dfrac{Ix[F, G] \qquad F^x_{c_1} \qquad F^x_{c_2}}{c_1 = c_2}$$

The Rigid Parameters Rules

Do we need to add a rule corresponding to Garson's $(\exists i)$ for picking rigid designators for complex terms?

$$\begin{array}{c} [a = t]^i \\ \Pi \\ (\exists i) \quad \dfrac{A}{A}\, i \end{array}$$

where a is a parameter that does not occur free in A, t nor any undischarged assumptions of Π except $a = t$.

No. In fact it is worse, and we mustn't, as the analogous rule:

$$\begin{array}{c} [Ix[B, a = x]]^i \\ \Pi \\ \dfrac{A}{A}\, i \end{array}$$

is inconsistent. Let B be a contraction $\bot(x)$. As $Ix[\bot(x), G] \vdash \bot$, for any G, this rule trivialises the system.

Semantics for I

$\mathcal{M}, w \Vdash_v Ix[A, B]$ iff there is a $d \in \mathcal{D}$ such that for the x-variant v' of v such that $v'(x) = d$, $\mathcal{M}, w \Vdash_{v'} A$, and for any x-variant v'' of v', if $\mathcal{M}, w \Vdash_{v''} A$, then $v''(x) = d$, and $\mathcal{M}, w \Vdash_{v'} B$.

So: $Ix[F, G]$ is true at a world w iff F is assigned a singleton subset of the domain of the model at w and its element is G at w.

Note: I haven't proved completeness, but I think this works.

Uniqueness of F in the domain of the model is quite a strong requirement. If every $x \in \mathcal{D}$ is in some $\mathcal{D}(w)$, then for the universal possibility $Ix[F, G] \vDash Ix[F, \Diamond\exists!x]$. (Garson does not demand this).

An alternative would be to assign to each world an outer domain that is a subset of the domain of the model and require uniqueness only there. But this is not customary, and I'm not sure it would make a difference, as we haven't got outer quantifiers or a predicate true only of the things in the outer domain ('subsists').

Comparisons. I. Non-Modal. I

By ‘there is’ I’ll mean ‘there is in the outer domain’; by ‘exists’ I mean ‘there is in the inner domain’. For the comparisons, assume we add the binary quantifier to Garson’s system.

$Ix[A, B]$ neither implies nor is implied by $B(\imath xA)$

If there is no unique or more than one A, $Ix[A, B]$ is false, but $B(\imath xA)$ may be true. If no unique A exists, $Ix[A, B]$ may be true (if there is a unique A that is B), and $B(\imath xA)$ false.

$Ix[A, B]$ neither implies nor is implied by $\lambda xB(\imath xA)$

Scope distinctions do not matter in the non-modal language of Garson’s system: $\vdash B(\imath xA) \leftrightarrow \lambda xB(\imath xA)$.

Thus we cannot expect a straightforward translation between Garson’s system and mine.

Comparisons. II. Non-Modal. II

$B(\imath x A) \wedge \exists! \imath x A$ does not imply $Ix[A, B]$

If there is an A in addition to the existing one, $Ix[A, B]$ is false.

So: $\exists! \imath x A$ does not imply $Ix[A, \exists! x]$

But: $Ix[A, \exists! x]$ implies $\exists! \imath x A$

Because then the unique A exists, so $\imath x A$ is that unique existing A. For the same reason we have:

$Ix[A, B] \wedge \exists! \imath x A$ implies $B(\imath x A)$

In the modal case B needs to be a predication.

Finally: $\exists! \imath x A \wedge B(\imath x A)$ is equivalent to $Ix[A \wedge \exists! x, B]$

Comparisons. III. Variations of Leibniz' Law

In my system, Leibniz' Law is not applicable to definite descriptions, as they are not terms, but its effect can be mimicked:

$$c = \imath x D, A^x_{\imath x D} \vdash A^x_c \leftrightsquigarrow Ix[D, c = x], Ix[D, A] \vdash A^x_c$$

$$c = \imath x D, A^x_c \vdash A^x_{\imath x D} \leftrightsquigarrow Ix[D, c = x], A^x_c \vdash Ix[D, A]$$

$$\imath x D = \imath x E, A^x_{\imath x D} \vdash A^x_{\imath x E} \leftrightsquigarrow Ix[D, Iy[E, x = y]], Ix[D, A] \vdash Ix[E, A]$$

In fact, only the last is needed, because $\vdash A^x_c \leftrightarrow Ix[x = c, A]$.

$\leftrightsquigarrow$ evidently does not mean 'translates as'.

On the right of $\leftrightsquigarrow$, x may be in the scope of a modal operator in A (because we're talking *de re* of the D that is A).

In Garson's system: For any formula A, if x is not in the scope of $\Box$ in A or if $s = t \vdash \Box\, s = t$, then $s = t, A^x_s \vdash A^x_t$.

Comparisons. IV. Universal Instantiation with Descriptions

In Garson's system $(\forall E)$ can be generalised:

If x is not in the scope of $\Box$ in A or if t is rigid: $\forall x A, \exists ! t \vdash A^x_t$.

Rigidity can be weakened. It suffices that for some parameter a not occurring in A or t, $a = t \vdash \Box\ a = t$.

If t is rigid, this is always the case: take a fresh parameter, assume $a = t$, infer $\Box\ a = t$, eventually use $(\exists i)$ to discharge $a = t$.

$\forall x A, Ix[D, \exists ! x] \vdash Ix[D, A]$ is derivable, without restrictions, as the description is *de dicto*.

Comparisons. V. Rigidity. I

A term is rigid in w iff it picks out the same object in every world accessible from w. What does rigidity mean in my system?

$Ix[A, B]$ is about the A rigidly if the same object that is the A that is B in w is the A in every accessible world. Thus 'The A is necessarily the A' expresses the rigidity of 'the A' in terms of binary quantification. Let's restrict A to predicate letters. Then:

$\mathcal{M}, w \Vdash_v Ix[A, \Box Iy[A, x = y]]$: $I(A, w)$ is a singleton $\{a\}$, and in every accessible world w', $I(A, w') = \{a\}$.

$\mathcal{M}, w \Vdash_v \Box Ix[A, x = c]$: the interpretation of A is the singleton $\{I(c)\}$ in every accessible world (constants are rigid).

$\mathcal{M}, w \Vdash_v \Box Ix[A, \top]$: the interpretation of A is a singleton set in every accessible world.

Note: $A^x_c \to \Box A^x_c, \Box Ix[A, \top] \vdash \Box Ix[A, x = c]$.

Comparisons. VI. Rigidity. II

Let's say that if $\mathcal{M}, w \Vdash_v Ix[A, \Box Iy[A, x = y]]$, then A is rigidly unique at w.

A version of the necessity of identity follows:

$$Ix[A, \Box Iy[A, x = y]], Ix[A, x = c] \vdash \Box Ix[A, x = c]$$

If A is rigidly unique, then if the A is c, this is necessarily so.

Rigid uniqueness suffices for a version of the necessity of difference:

$$Ix[A, \Box Iy[A, x = y]], Ix[A, x \neq c] \vdash \Box Ix[A, x \neq c]$$

If A is rigidly unique, then if the A is different from c, this is necessarily so.

Comparisons. VII. Rigidity. III

In Garson's system, scope distinction can be ignored for rigid terms. In fact, something slightly weaker suffices. As constants are rigid, the below is interesting only when t is a definite description.

1. For any formula B and term t, if x is not in the scope of $\Box$ in B or if for some parameter a not occurring in B or t, $a = t \vdash \Box\ a = t$, then $\vdash \lambda x B(t) \leftrightarrow B^x_t$.

2. For any formula B and term t, if x is not in the scope of $\Box$ in B or if for some parameter a not occurring in B, t, $a = t \vdash \Box\ a = t$, then $\vdash \lambda x \Box B(t) \leftrightarrow \Box B^x_t$ and $\vdash \Box B^x_t \leftrightarrow \Box \lambda x B(t)$.

Nothing corresponds to 1. in my system, as DD always require scope markers. But we have: $\vdash Ix[Iy[A^x_y, x = y], B] \leftrightarrow Ix[A, B]$.

$\vdash \lambda x \Box B(t) \leftrightarrow \Box \lambda x B(t)$ of 2. corresponds to:

$Ix[A, \Box Iy[A, x = y]] \vdash Ix[A, \Box B] \leftrightarrow \Box Ix[A, B]$

A Few Words on Fitting and Mendelsohn. I

$\mathcal{I}(\imath x A, w)$ is defined in $\mathcal{M}$ under valuation v iff there is exactly one x-variant v' of v such that $\mathcal{M}, w \Vdash_{v'} A$. Then $\imath x A$ designates at w under v and $\mathcal{I}(\imath x A, w) = v'(x)$. If $\imath x A$ designates at w under valuation v, then $\mathcal{M}, w \Vdash_v \lambda x B(\imath x A)$ iff $\mathcal{M}, w \Vdash_{v'} B$, where v' is the x-variant of v such that $v'(x) = \mathcal{I}(\imath x A, w)$. If $\imath x A$ does not designate at w under valuation v, then $\mathcal{M}, w \nVdash_v \lambda x B(\imath x A)$.

Suppose $\imath x A$ designates at w under v. Then $(*)$ there is a $d \in \mathcal{D}$, i.e. $\mathcal{I}(\imath x A, w)$, s.th. there is exactly one x-variant v' of v s.th. $v'(x) = d$ and $\mathcal{M}, w \Vdash_{v'} A$. Then if $\mathcal{M}, w \Vdash_v \lambda x B(\imath x A)$, then $\mathcal{M}, w \Vdash_{v'} B$, and so $\mathcal{M}, w \Vdash_v Ix[A, B]$.

If $\imath x A$ does not designate at w under v, then $(*)$ fails, and so $\mathcal{M}, w \nVdash_v \lambda x B(\imath x A)$ and also $\mathcal{M}, w \nVdash_v Ix[A, B]$.

So either way, $\mathcal{M}, w \Vdash_v \lambda x B(\imath x A)$ iff $\mathcal{M}, w \Vdash_v Ix[A, B]$.

A Few Words on Fitting and Mendelsohn. II

Although $\mathcal{M}, w \Vdash_v \lambda x B(\imath x A)$ iff $\mathcal{M}, w \Vdash_v Ix[A, B]$, there are important differences between the two approaches:

(1) They use a different system of free logic, in which $\models \exists x \exists! x$: its valid closed formulas without predicate abstracts are those of classical logic. Thus at every world at least one object exists.

(2) Constants need not be rigid. Accordingly forming formulas with constants requires scope distinctions, i.e. λ abstracts: where A is a predicate letter, $\lambda x A(c)$ is a formula, A^x_c is not. Atomic formulas are only those formed from predicate letters by variables (or parameters). (Hence the valid closed formulas without predicate abstracts also do not contain constants.)

(3) Constants may not designate. If c does not designate, $w \nVdash_v \lambda x B(c)$, and so $\not\models \lambda x(x = x)(c)$, i.e. the formula expressing the law of identity for constants is not valid. (Officially neither is $a = a$, a a parameter, as it is not a closed formula.)

A Few Words on Fitting and Mendelsohn. III

The differences of the last slide can, I think, be overcome:
(1) Add $\exists x \exists ! x$ as an axiom.
(2) Restrict the language: atomic formulas are formed only with variables/parameters, constants require the binary quantifier: if c is a constant, A a formula, $Ix[x = c, A]$ is a formula. (NB: $x = c$ is not a formula in Fitting and Mendelsohn's system.)
(3) Restrict $(\forall E)$, $(= I)$, $(= E)$, (RC), (II), (IE_1), (IE_2) to parameters, except when A, F, G have the form $Iy[y = x, B]$.
(4) Restrict undischarged premises and conclusions of deductions to closed formulas. ($a = a$, a a parameter, is permitted as premise.)
(5) Translate formulas with λ as follows:

$$\tau(\lambda x A(c)) = Ix[x = c, A] \text{ and } \tau(\lambda x A(\imath y B)) = Iz[A^x_z, B^y_z], \ z \text{ fresh.}$$

These are first steps towards establishing equivalence, but more work is needed. The formula $Ix[x = c, x = x]$ expressing the law of identity is not provable. It shouldn't be if c does not designate. If constants always designate, we may need to add it as an axiom.

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