Non-Classical Logics. Theory and Applications

Short presentations

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Contents

A Case for Weak Kleene ST

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Keywords: Strict/Tolerant Logic, Weak Kleene, Paradox, Revenge, Sorites.

The substructural Strict/Tolerant logic based on a strong Kleene valuation (sST) was motivated by its ability to express a fully transparent truth predicate and the tolerance principle without falling into the traps of semantic and soritical paradoxes. Even though sST rejects the meta-inferential rule of Cut, it has been shown that many instances of Cut are recoverable. Thus, not only theories of truth and vagueness based on sST can avoid the semantic and soritical paradoxes, these theories stay very close to classical theories which is counted as a virtue of sST. In a recent paper by Murzi and Rossi, the authors argue that the notion of (un)paradoxicality plays a major role in recapturing the "safe" instances of Cut. However, the theory of truth based on sST cannot be extended to express the notion (un)paradoxicality on pain of revenge paradox. Similarly, in a recent paper by Bruni and Rossi, the authors argue that the theory of vagueness based on sST cannot be extended to express the notion of determinateness on pain of revenge paradox, even though "determinateness" plays a major role in the theory.

In this paper, we argue that given the analysis of these revenge paradoxes, the Strict/Tolerant logician should prefer a weak Kleene variation of the Strict/Tolerant logic (wST). We argue that wST can express a fully transparent truth predicate and the tolerance principle as well as the notions of (un)paradoxicality and determinateness (though we prefer to use the notion of groundedness to encompass both of these notions), while still being immune to revenge. We conclude that the logic wST is more appealing than sST, for it has the same virtues as sST while it has an unmatched expressive power.

Reading Newton's *De Analysi* by hyperfinite sums

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Keywords: Newton, the fundamental theorem of calculus, indivisibles, infinitesimals, hyperfinite sum.

In the 1669 manuscript *De Analysi*, Newton derives three theorems, now cornerstones of modern calculus: the power series for arcsine, the power series for sine, and the area under the curve $y(x) = x^{\frac{m}{n}}$ equals $\frac{n}{m+n}x^{\frac{m+n}{n}}$ (Rule I). He also sets the rule stating the area under finitely or infinitely many curves equals the sum of areas under each curve (Rule II) and shows how to expand into a power series then-standard functions such as $\frac{a^2}{b+x}$ or $\sqrt{a^2+x^2}$ based on some algebraic laws (Rule III).

Generally, Newtons's approach in Rule I hinges on an odd procedure of summing up infinitesimal area moments; for the series of arcsine, he combines Rule II applied to infinitesimal arc moments with the same peculiar sum operation; dealing with area and arc moments, he employs the concept of the infinitely small unit segment (*indivisible*); deriving the sine series, he adopts an approach other than Rules I to III.

Specifically, proving Rule I, Newton shows that if $z(x) = \frac{n}{m+n}x^{\frac{m+n}{n}}$ is a function of an area under a curve, then the curve is $y(x) = x^{\frac{m}{n}}$. At the end, he writes "Conversly therefore if $ax^{\frac{m}{n}} = y$, then will be $\frac{n}{m+n}ax^{\frac{m+n}{n}} = z$."

We interpret Newton's Rule I with techniques of nonstandard analysis and represent his arguments on a hyperfinite grid, essentially a discrete domain instead of a continuous one. Bridging the gap between finite and infinite, we mimic Newton's approach, define the area under a curve as a hyperfinite sum, and prove the unproved part of Rule I using 17th-century techniques.

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Bridging Classical and Modern Approaches to Thales' Theorem: From Euclidean Proportion to Automated Theorem Proving

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Keywords: foundations of geometry, Thales theorem, proportion, automatic proves, GCLC.

This paper bridges the classical and modern approaches to Thales' theorem. We start with commensurable lines and then introduce the general version. While the commensurable version relies on elementary techniques, the general one requires the arithmetic of line segments or real numbers. We reference Euclid's proof of Thales' theorem (Elements, VI.2) and consider its implications for automatic theorem proving.

Thales' theorem, also known as the intercept or fundamental theorem of proportionality, is a crucial topic that distinguishes Euclid's methodology from modern axiomatic systems. In the Elements, it is a part of Book VI and builds on proportion developed in Book V – a technique replaced in the 20th century by the arithmetic of real numbers [2]. Its proof relies on proposition VI.1 and refers to the definition of proportion (V.def.5). Euclidean proportion relates pairs of figures of the same kind, such as line segments, triangles, and angles. At the same time, modern systems define proportion only for line segments.

We review the proofs of Thales' theorem throughout the 20th-century geometric systems, significantly those developed by Hilbert [6], Hartshorne [4], Birkhoff [1], Millman & Parker [8], Borsuk & Szmielew [3], and Schwabhäuser, Szmielew & Tarski [9]. These systems highlight the challenge of proving Thales' theorem, with two main strategies identified: segment arithmetic (Hilbert, Hartshorne, Schwabhäuser, Szmielew, Tarski) and the implementation of real numbers into geometric systems ([3], [8]). The latter includes either axioms guaranteeing the existence of a bijection between real numbers and a straight line [1] or deriving this bijection from Hilbert-style axioms [3]. We examine proofs using segment arithmetic and discuss Borsuk and Szmielew's measure theorem and Millman and Parker's proof. Additionally, we consider Birkhoff's axiom on the relationship between real numbers and a straight line.

Modern systems that bypass Euclid's proof of Thales' theorem introduce real numbers or segment arithmetic, resulting in complex proofs. Euclid's proof is straightforward but uses techniques abandoned by modern mathematics. We propose an intermediary solution: an axiomatic theory, making proposition VI.1 an axiom, enabling the recovery of the original proof.

We present an approach to Thales' theorem based on the axioms of the area method and automatic theorem proving in the GCLC system developed by Chou, Gao, Zhang [5] and Janicic, Narboux, Quaresma [7].

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A nonclassical theory of sets and functions: advancing the foundations of mathematics

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Keywords: foundations of mathematics, nonclassical logic, finite axiomatizations.

It is well-known that ZF, the most widely used foundation for mathematics, has two pathological features, which Von Neumann and Zermelo himself were not happy with:

(i) ZF has infinitely many axioms;

(ii) if ZF has a model, it has a countable model.

It is also well-known, in fact proven, that this cannot be fixed in the framework of classical first-order logic.

In [1], a finitely axiomatized nonclassical theory $\mathfrak{T}$ of sets and functions, which incorporates category theory and axiomatic set theory, has been introduced as the collection of axioms one has to accept to get rid of the features (i) and (ii). The theory $\mathfrak{T}$ consists of some twenty axioms that can be reformulated as simple theorems of ZF, plus one nonclassical axiom, which is called the 'sum function axiom': this sum function axiom is a new mathematical principle, which is so powerful that it allows the derivation of the infinite schemes of separation and replacement of ZF from $\mathfrak{T}$—with the help of nonclassical rules of inference, that is. Due to the definition of validity of nonclassical formulas in a model M of $\mathfrak{T}$, the downward Loewenheim-Skolem theorem does not hold; consequently, if $\mathfrak{T}$ has a model, it doesn't have a countable model. And just now it has been proven that $\mathfrak{T}$ is relatively consistent with ZF, demonstrating that $\mathfrak{T}$ is an advancement in the foundations of mathematics [2]. Of course one may doubt that nonclassical mathematics will ever become mainstream in everyday mathematical practice, but the thing here is that the sum function axiom can stay in the background: for everyday mathematical practice it suffices to use the standard first-order theorems of $\mathfrak{T}$.

The purpose of this talk is to give a general overview of this research program. An attempt will be made to explain the nonclassical sum function axiom to the audience in the available time slot, using Venn diagrams.

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There are unknown intuitionistic truths (at least in some intuitionistic models)

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Keywords: intuitionistic logic, epistemic logic, knowability paradox.

Developing intuitionistic logic with the notion of constructive knowledge gives rise to plethora of controversies. At the heart of the dispute lies the problem of specifying relations between notions of *constructive truth* and *constructive knowledge.* At the level of propositional logic it boils down to rejection or acceptance of the following reflection principles:

(r) $K\phi \supset \phi$ (*reflection*)

(cr) $\phi \supset K\phi$ (*co-reflection*)

The co-reflection principle is controversial: some authors rejects it (see [4] and [3]), but some, using brilliant arguments, accepts it (see [1]), while rejecting (r). The acceptance of co-reflection has far-reaching consequences. Recall positive ($K\phi \supset KK\phi$) and negative ($\neg K\phi \supset K\neg K\phi$) introspection principles hold (by substitution). From our perspective, the original sin of co-reflection is that it makes the formula:

(cl) $\exists\phi(\phi \wedge \neg K\phi)$

expressing the classical notion of knowledge, inconsistent. (cl) plays a similar role as the law of excluded middle — the latter says something important about classical notion of truth while the former specify the classical notion of knowledge. We believe that specifically classical principles should not be inconsistent in intuitionistic logic. That does not mean we have to give to them our assent.

The aim of our talk is to discuss arguments concerning the acceptance or rejection of (r) and (cr), specifying the underlying assumptions. We believe there are decent reasons to rejects the validity of the latter in intuitionistic logic of knowledge. We show how such a knowledge operator can be defined in Kripke semantics. We will discuss the prospects of solving the knowability paradox ([2]) in the proposed framework.

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Intuitionistic K: Proofs, Countermodels and Simulation

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Keywords: nested sequents, proof search, countermodels.

The logic **IK** is the intuitionistic variant of normal modal logic **K**, introduced by Fischer Servi [1], Plotkin and Stirling [4], and studied by Simpson [5]. This logic is considered a fundamental intuitionstic modal system as it corresponds, modulo the standard translation, to a fragment of intuitionstic first-order logic. The language of **IK** is generated from constants $\top$, $\bot$ and a set At of propositional variables and by means of the operators $\wedge$, $\vee$, $\supset$, $\Box$ and $\Diamond$. The semantics of **IK** is defined in terms of bi-relational models, consisting of a pre-order relation $\leq$ from intuitionistic models and a binary relation R corresponding to the accessibility relation of modal Kripke frames. Bi-relational models additionally satisfy the frame conditions of *forward* and *backward confluence*, governing the interaction between the two relations.

Definition. A *bi-relational model* is a quadruple $\mathcal{M} = (W, \leq, R, V)$ where W is a nonempty set of elements, called *worlds*, $\leq$ is a reflexive and transitive relation (a *pre-order*) over W, R is a binary relation over W and the valuation function $V : W \longrightarrow \wp(\mathsf{At})$ satisfies the following *hereditary condition*: For all $x, y \in W$, if $x \leq y$ then $V(x) \subseteq V(y)$.

Moreover, $\mathcal{M}$ satisfies the frame conditions of *forward* and *backward confluence*:

(FC) For all $x, x', z \in W$, if $x \leq x'$ and xRz, there is $z' \in W$ s.t. $x'Rz'$ and $z \leq z'$.

(BC) For all $x, z, z' \in W$, if xRz and $z \leq z'$, there is $x' \in W$ s.t. $x'Rz'$ and $x \leq x'$.

Various proof systems for **IK** have been proposed in the literature, among which nested and labelled sequent calculi [2, 3, 6]. We introduce an innovative labelled-free bi-nested sequent calculus for **IK**, which is called $\mathbf{C_{IK}}$. This proof system comprises two kinds of nesting structures: $\langle \cdot \rangle$, corresponding to the pre-order relation of bi-relational models for **IK**, and $[\cdot]$, corresponding to the accessibility relation. The frame conditions of bi-relational models are modularly captured in the proof system by the 'interaction rules', ($\text{inter}_{\mathsf{fc}}$) and ($\text{inter}_{\mathsf{bc}}$), operating on the two nestings. The system is given in Figure 1.

$$\frac{}{G\{\Gamma, \bot \Rightarrow \Delta\}}\ (\bot_L) \qquad \frac{}{G\{\Gamma \Rightarrow \top, \Delta\}}\ (\top_R) \qquad \frac{}{G\{\Gamma, p \Rightarrow \Delta, p\}}\ (\mathrm{id})$$

$$\frac{G\{A, B, \Gamma \Rightarrow \Delta\}}{G\{A \wedge B, \Gamma \Rightarrow \Delta\}}\ (\wedge_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, A\} \quad G\{\Gamma \Rightarrow \Delta, B\}}{G\{\Gamma \Rightarrow \Delta, A \wedge B\}}\ (\wedge_R)$$

$$\frac{G\{\Gamma, A \Rightarrow \Delta\} \quad G\{\Gamma, B \Rightarrow \Delta\}}{G\{\Gamma, A \vee B \Rightarrow \Delta\}}\ (\vee_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, A, B\}}{G\{\Gamma \Rightarrow \Delta, A \vee B\}}\ (\vee_R)$$

$$\frac{G\{\Gamma, A \supset B \Rightarrow A, \Delta\} \quad G\{\Gamma, B \Rightarrow \Delta\}}{G\{\Gamma, A \supset B \Rightarrow \Delta\}}\ (\supset_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, \langle A \Rightarrow B\rangle\}}{G\{\Gamma \Rightarrow \Delta, A \supset B\}}\ (\supset_R)$$

$$\frac{G\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]\}}{G\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}\ (\Box_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, \langle \Rightarrow [\Rightarrow A]\rangle\}}{G\{\Gamma, \Rightarrow \Delta, \Box A\}}\ (\Box_R)$$

$$\frac{G\{\Gamma \Rightarrow \Delta, [A \Rightarrow]\}}{G\{\Gamma, \Diamond A \Rightarrow \Delta\}}\ (\Diamond_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, \Diamond A, [\Sigma \Rightarrow \Pi, A]\}}{G\{\Gamma \Rightarrow \Delta, \Diamond A, [\Sigma \Rightarrow \Pi]\}}\ (\Diamond_R)$$

$$\frac{G\{\Gamma, \Gamma' \Rightarrow \Delta, \langle \Gamma', \Sigma \Rightarrow \Pi\rangle\}}{G\{\Gamma, \Gamma' \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle\}}\ (\mathrm{trans})$$

$$\frac{G\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi, [\Lambda \Rightarrow \Theta^*]\rangle, [\Lambda \Rightarrow \Theta]\}}{G\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle, [\Lambda \Rightarrow \Theta]\}}\ (\mathrm{inter}_{\mathsf{fc}})$$

$$\frac{G\{\Gamma \Rightarrow \Delta, [\Lambda \Rightarrow \Theta, \langle \Sigma \Rightarrow \Pi\rangle], \langle \Rightarrow [\Sigma \Rightarrow \Pi]\rangle\}}{G\{\Gamma \Rightarrow \Delta, [\Lambda \Rightarrow \Theta, \langle \Sigma \Rightarrow \Pi\rangle]\}}\ (\mathrm{inter}_{\mathsf{bc}})$$

Figure 1: Rules of $\mathbf{C_{IK}}$

Theorem 1 (Soundness)**.** If a sequent is provable in $\mathbf{C_{IK}}$, then it is valid.

Then we introduce a decision algorithm for **IK**, that is, an algorithm that for any formula decides whether it is valid or not. The algorithm implements a terminating proof search strategy in a cumulative and set-based version of $\mathbf{C_{IK}}$ which is called $\mathbf{CC_{IK}}$.

Proposition 2. Let S be a sequent. S is provable in $\mathbf{C_{IK}}$ if and only if it is provable in $\mathbf{CC_{IK}}$.

Theorem 3 (Termination)**.** Let A be a formula. Proof-search for the sequent $\Rightarrow A$ terminates and it yields either a proof of A or a finite derivation where all the non-axiomatic leaves are global-saturated.

Next, we show that $\mathbf{C_{IK}}$ allows for direct counter-model extraction: from a single failed derivation, it is possible to construct a finite counter-model for the formula at the root. This can be seen as the main advantage of our calculus over other labelled-free calculi. In order to extract a model from a global-saturated sequent, however, we need to additionally keep track of specific components occurring in set-based sequents. To this aim, we shall define an annotated version of $\mathbf{CC_{IK}}$, called $\mathbf{CC^{n}_{IK}}$, which operates on sequents whose components are decorated by natural numbers, the *annotations*, and are equipped with an additional structure to keep in memory a binary relation on such annotations.

The finite countermodel extraction from failed proof search entails completeness.

Theorem 4 (Completeness)**.** If A is valid in **IK**, it is provable in $\mathbf{CC}^{\mathrm{n}}_{\mathbf{IK}}$, and thus in $\mathbf{CC}_{\mathbf{IK}}$.

Lastly, we compare $\mathbf{C}_{\mathbf{IK}}$ with a fully labelled sequent calculus $\mathsf{labIK}_{\leq}$ (cf. [2]). We simulate any derivation (rooted by a formula) in $\mathsf{labIK}_{\leq}$ into one in $\mathbf{C}_{\mathbf{IK}}$. The whole procedure is divided into three steps, (i) $\mathsf{labIK}_{\leq}$ to $\mathbf{CC}^{\mathrm{n}}_{\mathbf{IK}}$; (ii) $\mathbf{CC}^{\mathrm{n}}_{\mathbf{IK}}$ to $\mathbf{CC}_{\mathbf{IK}}$; and lastly (iii) $\mathbf{CC}_{\mathbf{IK}}$ to $\mathbf{C}_{\mathbf{IK}}$. Steps (ii) and (iii) are trivial; for (i) we prove the following result:

Proposition 5. For each formula, a derivation in $\mathsf{labIK}_{\leq}$ can be simulated by one in $\mathbf{CC}_{\mathbf{IK}}$.

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Weak Extensions and Free Logic: a Proof-Theoretic Case Study

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Keywords: weak extension, positive free logic, virtual term, virtual domain.

We may observe in mathematical praxis an interesting phenomenon: a statement is understood as a statement over a small domain, but its formulation uses notions only available in an extension of this domain. We take as paradigmatic example the statement that the Gauss numbers satisfy the following equation:

$$\sum_{k=0}^{n} k = \frac{1}{2} n(n+1)$$

This is clearly a statement about natural numbers. But the statement depends on a recourse to fractions and rational numbers. Consequently, we find a comment in the proof that the right side of the equation is, indeed, a natural number, as one of both n or $n+1$ must be even. We call this phenomenon, in which the underlying domain is only apparently transcended, a *weak extension.* A more elaborate example would be the closed-form representation of the Fibonacci numbers via Binet's formula using fractions and irrational numbers.

It is natural to ask how weak extensions are adequately represented in formal logic.[1] Such a representation has to satisfy, in the case of Gauss numbers, the following requirements:

1. The domain of discourse remains the natural numbers axiomatised in (a suitable version of) Peano arithmetic (PA), quantifier and variables range over the natural numbers. In particular, the ring of rationals is not constructed.

2. There are term-forming operations available, the fractions, which do not denote necessarily in the domain of discourse. Additionally, denotation has to be expressible in the formal language.

For this purpose, we develop a *free logics*, the *logic of virtual extensions*, extending classical logic by permitting non-denoting terms. More precisely, using the terminology found in Nolt [2], we suggest a positive, bivalent and non-inclusive free logic. Subsequently, we sketch some details:

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[1] I am thankful to Reinhard Kahle actually asking this question.

The Grammar. Besides the standard symbols of first-order logic (including non-logical symbols of PA), we need a unary predicate symbol $!\cdot$ expressing denotation and a *virtual function symbol* $\cdot/\cdot$ representing fractions. The fist-order terms are defined as expected, but with the restriction that virtual function symbols may only be applied on standard terms (not containing fractions); furthermore, we do not demand that all standard operations are applicable on virtual terms containing fractions. The formulae are defined as expected.

The Calculus. In order to obtain a suitable calculus, we modify the calculus of Natural Calculation [1] (a convenient variant of Natural Deduction in which equations are proved by calculations) as follows:

- New inference rules expressing that all *standard* terms denote. In our example based on PA:

$$\frac{}{!x} \quad ; \quad \frac{}{!0} \quad ; \quad \frac{!t}{!S(t)} \quad ; \quad \frac{!t \qquad !s}{!(t+s)} \quad ; \quad \frac{!t \qquad !s}{!(t \cdot s)}$$

- A new inference rule permitting to infer non-denotation of some virtual terms:

$$\frac{\forall x. x \neq t}{\neg !t} \quad \text{where } x \text{ not free in } t.$$

- Modification of the elimination of the universal quantifier rule by adding an additional premise stating that the term intended for substitution denotes; no modification of the introduction rule.

$$\frac{\forall x. A(x) \qquad !t}{A(t)} \quad \text{where } t \text{ free for substitution in } A(x) \text{ and } A(t) \text{ well-formed.}$$

  Analogously, with respect to the existential quantifier rules.

A meaningful use of the virtual function symbols and the new predicate $!\cdot$ requires, additionally, the extension of the underlying formal theory. Such a *virtual extensions* contains, in our example case, axioms characterising the equality of fractions (as $\forall x.\, x/1 = x$).

The Semantics. The virtual extension of a standard structure $\langle\, D, \Omega \,\rangle$ is based on a virtual domain $V(D)$, which is a set of closed terms containing (1) names $\overline{d}$ for all real objects $d \in D$ and (2) complex terms generated out of these names according to the term definition (obeying to the restrictions for virtual terms). Hence, $\overline{1}$, $\overline{1}/\overline{1}$ and $\overline{1}/\overline{2}$ are elements of the virtual domain in our example, the first two denoting (in $\mathbb{N}$) and the last one not denoting. The virtual character of the elements of $V(D)$ is reflected by the fact that equality is not interpreted by actual identity of these terms (or of equivalence classes of them), but rather via a suitable congruence relation $\sim$ on $V(D)$. In this setting, $!t$ is true, if the evaluation of t in $V(D)$ is congruent with a name $\overline{d}$ of a real object d. (Hence, $!\overline{1}/\overline{1}$ is true, as $\overline{1}/\overline{1} \sim \overline{1}$ and $1 \in \mathbb{N}$.)

The conception of our semantics can be seen as a two-domain solution, where the real domain is embedded into the outer domain (but not contained). Our semantics seems different to those discussed by Nolt [2]. We consider the suggested semantics ontologically parsimonious and very natural, at least from the point of view of a mathematician. It is easily seen that virtual structures do not change the evaluation of standard formulae, which means that virtual extensions of formal theories are conservative. Soundness and completeness of the calculus are work in progress.

Conclusion. The presented logic seems suitable to represent adequately weak extensions. In particular, we do not have to create, in our example, the ring of rationals (neither explicitly nor implicitly): we neither consider negative numbers nor fractions of fractions.

Virtual extensions seem to have relevance beyond the mentioned examples. There is a long lasting tradition in mathematics to doubt the real existence of new objects, whenever they have been introduced. This was the case with negative numbers, with the real numbers and the complex numbers. Virtual extensions can serve as the formal underpinning of such philosophical stances.

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Mapping Probability with Logic: First Order Models in Puzzle Solving

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Keywords: First Order Logic, Probabilistic puzzles, Finite Models.

Solving logic puzzles by modelling them in first order logic (FOL) is old hat in computer science [3]. For instance, Prover9 theorem prover and Mace4 model finder [4] have been applied to logical puzzles [2]. We show here how Mace4 can be used to solve probabilistic tasks, by formalising probabilistic puzzles in equational FOL. Two formalisations are needed: one theory for all models of the given puzzle, and a second theory for the favorable models. Then Mace4 - that computes all the interpretation models of a FOL theory - is called twice. First, it is asked to compute all the possible models $\mathcal{M}_p$. Second, the additional constraint is added, and Mace4 computes only favorabile models $\mathcal{M}_f$. Finally, the definition of probability is applied: the number of favorable models is divided by the number of possible models.

Here is an example for you: *($Puzzle_1$) Let two decks of cards. Turn over the top card on each deck. What is the probability that at least one of those cards is the queen of hearts?* $Puzzle_1$ is formalised in Listing 1. The domain size is set to the first 52 natural numbers: i.e., all the functions from the theory are mapped against integer values from the interval [0..51]. With `max_models=-1`, we are asking for all interpretations of the FOL theory. Let the functions $deck_1$ and $deck_2$ take two values a and b from the domain. Mace4 computes $\mathcal{M}_p$=2,704 models, given by 52×52. Let queen of hearts be the value 7 (or another value) from the deck [0..51]. By adding the constraint in line 2 from Listing 2, Mace4 returns $\mathcal{M}_f$=103 models. Hence, the probability that at least one of those cards is the queen of hearts is $\frac{103}{2{,}704}$. This modelling-based approach avoids two common logical faults: (i) not considering the case of extracting both queens of heart with the possible wrong answer $\frac{1}{52} + \frac{1}{52} = \frac{1}{26}$; (ii) considering ordered pairs, where pairs like (Qh, Kh) are different from (Kh, Qh), with the wrong answer $\frac{52+52}{52\times 52} = \frac{104}{2{,}704}$.

Listing 1: Possible models for two random cards

```
assign(domain_size,52).
assign(max_models, -1).
set(arithmetic).
formulas(two_decks).
  deck1 = a.  deck2 = b.
end_of_list.
```

Listing 2: Favorable models for at least one queen of hearts

```
formulas(Qh).
  a = 7 | b = 7.
end_of_list.
```

Listing 3: Possible models when throwing three dice

```
assign(domain_size,7).
assign(max_models,-1).
set(arithmetic).
formulas(three_dice).
 Dice1 != 0.  Dice2 != 0.  Dice3 != 0.
end_of_list.
```

The above method is easily applied to dice puzzles. Let $Puzzle_2$*: What is the probability of rolling three six-sided dice and getting a value greater than 7?.* Let $Dice_1$, $Dice_2$, and $Dice_3$ the three dice (Listing 3). We set a domain size of 7 and we limit the possible values to 1 and 6 with $Dice_1 \neq 0$, $Dice_2 \neq 0$ and $Dice_3 \neq 0$. For three variables in [1..6], Mace4 returns $6 \times 6 \times 6 = 216$ models, as expected. To compute the favourable cases, we add the constraint $Dice_1 + Dice_2 + Dice_3 > 7$,Mace4 returns 181 models. Hence the probability is $\frac{181}{216}$.

In Mace4, FOL contains predicates, functions (including arithmetic), quantifiers, which provides enough expressivity to formalise different puzzles, including those with given probabilities. Let $Puzzle_4$: *Your task is to select socks from a dark drawer. There are six socks, a mixture of black and white. If two socks a repicked, the chances that a white pair is drawn are 2/3. What are the chances that a black pair is drawn?* Since there are six socks, we set the domain size to 6. Let the function $s(x) = 1$ if the sock x is white and $s(x) = 0$ of the sock x is black. Let W the number of white socks. Thus $\sum_{i=0}^{5} s(i) = W$ (line 8 in Listing 4). The probability of drawing two white socks is: $W/6 * (W - 1)/5 = 2/3$. This is equivalent to $3 * W * (W - 1) = 6 * 5 * 2$ (line 9 in Listing 4). Mace4 computes 6 possible models (right part of Listing 4). We are interested in favorable models in which there are two socks $x \neq y$ such that $s(x) = Black$ and $sock(y) = Black$ (line 2 in Listing 5). For this constraint, Mace4 fails to compute any model. Hence, the probability of extracting two black socks is zero. That is because all six possible models contain only one black sock.

Listing 4: Possible models for socks

```
assign(domain_size,6).
assign(max_models,-1).
set(arithmetic).
formulas(socks_all).
 White = 1. Black = 0.
 s(x) = White | s(x) = Black.
 s(0) + s(1) + s(2) + s(3) + s(4) + s(5) =W.
 3 * W * (W + -1) =  6 * 5 * 2.
end_of_list.
```

Model	$s(x)$
	0 1 2 3 4 5
1	1 1 1 1 1 0
2	1 1 1 1 0 1
3	1 1 1 0 1 1
4	1 1 0 1 1 1
5	1 0 1 1 1 1
6	0 1 1 1 1 1

Listing 5: Models of two black socks

```
exists x exists y (x != y & s(x) = Black & s(y) = Black).
```

The proposed approach equips students from the logic tribe to find the correct solution for puzzles from the probabilistic tribe, by using their favourite instruments: modelling and formalisation. With this method, the student is trained to approach the task in terms of possible models and favorable models. Since the definition of probability never fails, pitfalls usually leading to so many distinct results and apparently correct to the learners are avoided. One challenge is how far can one go with this approach. Ongoing work is investigating the limits of this method on various collections of probabilistic puzzles from or from popular brain teaser sites like `https://www.brainzilla.com/` or `https://www.mathsisfun.com/`, or collections including 11 probabilistic puzzles in [7], 56 puzzles in [5], [6] or the larger collection of dice problems of Matthew M. Conroy [1].

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Probability Logics for Reasoning About Measuring Quantum Observations on the spaces with infinite dimension

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Keywords: probability, quantum mechanics, measurement.

In [1] we presented families of probability logics suitable for reasoning about quantum observations on the spaces with finite dimension. Since quantum systems are often associated by infinite dimensional Hilbert spaces, we have extended this approach and we developed the logic QLP_{inf}, so that we can consider measurements on infinite dimensional spaces as well. The basic "quantum logic statement" we are considering is:"By measuring some observable, let's say O, we obtained its eigenvalue a". In order to express it we use modal formulas of the form $\Box\Diamond\alpha$. Using modal formula $\Box\Diamond\alpha$ instead of propositional formula α was introduced to distinguish the concepts: "Something is true" (which we denote by α) and "Something is oberved to be true, i.e., it is measured" (denoted by $\Box\Diamond\alpha$). Applying this approach and relying on the fact that $\Box$ does not distribute over $\vee$, we do not need non-distributive structures (like non-distributive lattices with numerous axioms and rules which are normally used in quantum logics). In [1] we gave an example and a detailed discussion of overcoming non-distributivity in this way. In quantum mechanics complex numbers are used for representation of waves. Thus the square of the modulus of complex numbers represents the probability of measuring certain values. Since the field of complex numbers is uncountable, in our semantics we use complex numbers with rational coordinates, i.e., numbers of the form $z = a + ib$ where $a, b \in \mathbf{Q}$. The set of formulas consists of modal formulas and probability formulas. Basic probability formula is a formula of the form

$$CS_{z_1,\rho_1;z_2,\rho_2;\ldots,z_j,\rho_j;\ldots}\Box\Diamond\alpha$$

and we can explain its meaning in the following way:

- Suppose that we have an observable O with infinitary many eigenvalues $a_1, a_2, \ldots$;
- Let $w_{1,1}, w_{1,2}, \ldots$ be eigenvectors that correspond to a_1 and assume that w is arbitrary vector and $c_{1,1}, c_{1,2}, \ldots \in \mathbf{C}$ are coefficients in linear representation of w in the system $w_{1,1}, w_{1,2}, \ldots$;

- Suppose that $\Box\Diamond\alpha$ means: "By measuring O we obtained a_1. Then $CS_{z_{1,1},\rho_{1,1};z_{1,2},\rho_{1,2};\ldots,z_{1,j},\rho_{1,j};\ldots}\Box\Diamond\alpha$ means that $z_{1,1}, z_{1,2}, \ldots$ approximate $c_{1,1}, c_{1,2}, \ldots$ in such way that $|z_{1,i} - c_{1,i}| \leq \rho_{1,i}$ where $||$ is a complex norm.

Formulas are interpreted in reflexive and symmetric Kripke models equipped with probability distributions over families of subsets of possible worlds that are orthocomplemented lattices. We give infinitary axiomatizations, prove the corresponding soundness and strong completeness theorems, and also decidability. In order to prove completeness, as expected, we proved Lindenbaum's theorem where we provided that in every maximal consistent set T, for every formula $\Box\Diamond\alpha$ and every $k \in \mathbf{N}$ there is a sequence $z_1^k, z_2^k \ldots$ such that $CS_{z_1^k,\frac{1}{k};z_2^k,\frac{1}{k};\ldots}\Box\Diamond\alpha \in T$. Then in the canonical model every world w is a maximal consistent set and $\mu(w, \Box\Diamond\alpha)) = (\lim_{k\to\infty} z_1^k, \lim_{k\to\infty} z_2^k, \ldots) = (c_1, c_2, \ldots)$ where $(c_1, c_2, \ldots)$ are required coefficients in linear representation of w. Finally, we mention that there is a significant difference in axiomatization (and thus in the parts that lead to proving the completeness theorem) between logics in [1] and the logic QLP_{inf}, in the sense that the rules of logic QLP_{inf} represent a significantly greater challenge.

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Continua of logcs for negation below in intuitionistic logic

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Keywords: Intuitionistic logic, Minimal logic, Subminimal logics, Constructive negation, Subminimal negation.

In this paper, we analyse the effects of the principle of explosion (PE) and inference rules related to PE within the context of subminimal logics which were introduced by Dimiter Vakarelov [9]. Since we have two backgrounds, we will first describe the backgrounds, and then clarify the aims of the paper.

Background (I) Intuitionistic logic contains PE. Minimal logic which is introduced by Ingebrigt Johansson in [5] is the logic excluding PE from Intuitionistic logic. When we treat classical, intuitionistic and minimal logics, negation is usually defined by making use of the absurdity constant and implication. We call this kind of negation *constructive negation*, by following the terminology used by Vakarelov in [9]. Using constructive negation, minimal logic is the weakest logic with respect to the strength of negation, on the assumption that the implication is at least intuitionistic. Hence no logic is weaker than minimal logic. When constructive negation is used, negation is too dependent on implication, and therefore insufficient for the purpose of analysing the effects of PE. Indeed, there are many other inference rules involving negation that cannot be discussed within the constructive negation approach.

There is another way to treat negation in classical/intuitionistic/minimal logics. That is to take negation as a *primitive* logical connective with one argument. We call this kind of negation *subminimal negation*, by following the terminology used Vakarelov in [9] again. In this way, we can define logics equivalent to classical/intuitionistic/minimal logics. Furthermore, we can define logics weaker than minimal logic, which we call subminimal logics.

In Almudena Colacito [2] and Satoru Niki [8], they considered logics weaker than minimal logic and found the hierarchy of them in subminimal logics. Furthermore soundness and completeness theorems for these subminimal logics were proven by using neighborhood semantics. However, since most of the logics discussed in [2, 8] do not contain PE, the effects of PE have been unclear.

Background (II) Vadim Yankov [11] proved, by using algebraic semantics, the existence of the continuum of logics between classical and intuitioistic logics. Considering the fact that there is a pair of logics having no logic between them, which is stated in Tsutomu Hosoi and Hiroakira Ono's survey [3], this result can be seen as indicating the big gap between classical and intuitionistic logics. Andrzej Wroński [10] improved Yankov's method to prove the existence of the continua of logics between pairs of other logics. Nick Bezhanishvili, Almudena Colacito and Dick de Jongh [1] proved, by using ideas of Yankov [11] and the neighborhood semantics, the existence of the continua of logics between some pairs of subminimal logics.

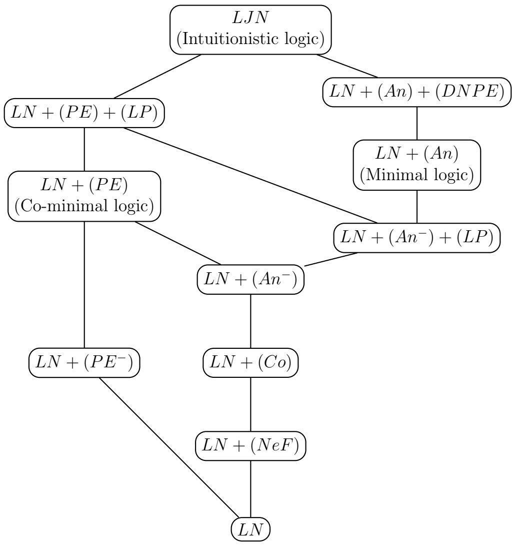

Figure 2: Logics in this paper

The aims of the paper In this paper, in order to examine PE in more detail, we add PE into subminimal logics and investigate its effects. As a result, we obtain a simpler characterization of co-minimal logic by Vakarelov [9] and the logic $LN + (PE^{-})$ in Figure 1 independent from subminimal logics which appeared in previous studies. Furthermore, in order to simplify the argument in [1], we improve Wroński's method in [10] to show the existence of the continua of logics between several pairs of subsystems of intuitionistic logic. This method can be applied also for pairs of logics discussed in Yankov [11] and Bezhanishvili et al. [1] and seems to have a more applications, because it uses algebraic semantics which is compatible with non-classical propositional logics in general.

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Proof theory for Epstein's logics of content relationship

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Keywords: logic of content relationship, natural deduction system, proof theory, relating logic, sequent system.

Logics of content relationship defined by Richard Epstein, i.e., relatedness logics and dependence logics, are special cases of relating logic (see [7, 8]). Such systems are extensions of classical logic obtained by introducing implications whose truth requires that:

- the antecedent is false or the consequent is true,
- the antecedent is related to the consequent with respect to content.

Epstein's systems differ due to various understandings of the content relationship. A detailed description of his systems alongside the philosophical motivation is given in the monograph [5, pp. 115–143] (cf. [4, 8, 9]). In the 1980s and 1990s, Epstein's logics, and some of their modifications, were described in various proof-theoretic ways. Aside from their axiomatic systems introduced by Epstein, Walter A. Carnielli [1] defined tableau systems for them. A different approach to tableau systems was presented in [6]. In [3], Luis Fariñas del Cerro and Valérie Lugardon defined sequent systems for certain modification of Epstein's dependence logics. In [2], logics of this kind were extended onto a first order language and sequent systems for these extensions were defined. Other interesting results were presented by Francesco Paoli, who focused on FDE-fragments of Epstein's logics. In [10], Paoli presented the FDEfragment of the relatedness logic S from the algebraic and axiomatic points of view. This analysis was extended onto other FDE-fragments of Epstein's logics in [11]. In that paper, Paoli also discussed tableaux for such FDE-fragments. In our paper, we would like to present Epstein's logics proof theory, focusing on approaches that have not been discussed so far. Thus, we will focus on natural deduction systems and comment on sequent systems.

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Hypersequent Calculi for Propositional Linear-Time Temporal Logic

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Keywords: propositional linear-time temporal logic, hypersequent calculus, temporal reasoning.

Propositional Linear-Time Temporal Logic

The syntax of Propositional Linear Temporal Logic (PLTL) extends that of Classical Propositional Calculus (CPC) by introducing future-time temporal operators. These temporal operators are: $\Box$ – 'always in the future'; $\Diamond$ – 'eventually in the future'; $\bigcirc$ – 'at the next moment'; and $\mathcal{U}$ – 'until'. This collection of operators is commonly called the *full set*, though it is often sufficient to use a reduced set, such as $\{\mathcal{U}, \bigcirc\}$, to define all other temporal operators.

A PLTL model consists of a discrete, linear sequence of states: $\mathcal{M} = s_0, s_1, s_2, \ldots$, which is isomorphic to the set of natural numbers, $\mathcal{N}$. Each state s_i, for $i \geq 0$, contains the propositions that are true at the i-th moment in time.

Axiomatic system for $PLTL$ following [5]:

$$
\begin{array}{l|l}
(K_\Box) & \vdash \Box(p \to q) \to (\Box p \to \Box q) \\
(K_\bigcirc) & \vdash \bigcirc(p \to q) \to (\bigcirc p \to \bigcirc q) \\
(FUNC) & \vdash \bigcirc\neg p \leftrightarrow \neg \bigcirc p \\
(FP_\Box) & \vdash \Box p \leftrightarrow (\bigcirc p \wedge \bigcirc\Box p) \\
(GFP_\Box) & \vdash (q \wedge \Box(q \to (p \wedge \bigcirc q) \to \Box p \\
(FP_\mathcal{U}) & \vdash p\mathcal{U}\, q \leftrightarrow (q \vee (p \wedge \bigcirc(p\mathcal{U}\, q))) \\
(LFP_\mathcal{U}) & \vdash \Box((q \vee (p \wedge \bigcirc r)) \to r) \to (p\mathcal{U}\, q \to r)
\end{array}
$$

PLTL is employed in the specification and verification of reactive and concurrent programs [8] [7, pp. 381-386], as well as in real-time programs [2,3], particularly for ensuring properties like *safety*, *liveness*, and *fairness*. Additionally, PLTL is used in temporal representation and reasoning [1,6, 10].

Hypersequent Calculus for PLTL

Propositional Linear-Time Temporal Logic (PLTL) has been considered in a number of proof systems, ranging from tableaux methods [8,11] and natural deduction [1,2] to sequent calculi [2,9]. The sequent calculi and tableaux methods approaches are based on the similar reasoning: usually involving state-prestate rules and facing similar problems around loop-generating formulas. Temporal connectives are handled by decomposing them into a requirement on the current state and the requirement on the rest of the sequence.

However, the labelled natural deduction approach employs a different rationale that involves relational judgements and, thus, faces different obstacles revolving around (the principle of) induction. We will present a labelled hypersequent calculi [4, 12] for PLTL that follows that relational reasoning and yet avoids the problem of induction. We choose the hypersequent calculus, rather than the standard sequent calculus, because it offers more expressive power, by enabling the additional transfer of information between different sequents.

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Socratic Proofs for Propositional Linear-Time Temporal Logic

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Keywords: propositional linear-time temporal logic, socratic proofs, inferential erotetic logic.

We will present a calculus of Socratic transformations for Propositional Linear-Time Temporal Logic (PLTL). Socratic transformations simulate reasoning aimed at solving a given logical problem by pure questioning [9, 13]. This is the method that transforms the initial question into consecutive questions without making any use of answers to the questions just transformed. Informally, a successful transformation, called a Socratic proof, ends with a question of a specified final form, which can be answered from a semantic point of view in only one rational way.

We aim to tackle temporal problems requiring specific — temporal — reasoning for their solution. That is why we developed our Socratic calculus based on Propositional Linear-Time Logic [4, 14], where formulae are interpreted over linear, discrete sequences of states that are finite in the past and infinite in the future.

Propositional Linear-Time Temporal Logic

The language of PLTL extends the language of Classical Propositional Calculus (CPC) by introducing future-time temporal operators. The range of these temporal operators that can be used to build the PLTL syntax consists of the following connectives: $\Box$ – 'always in the future' $\bigcirc$ –'at sometime in the future'; $\bigcirc$ – 'at the next moment in time'; $\mathcal{U}$ – 'until'. This set of temporal operators is often referred to as the *full set* while it is sufficient to use a reduced set, for example with only two of these - $\{\mathcal{U}, \bigcirc\}$ to express all other temporal operators.

A model for PLTL formulae is a discrete, linear sequence of states: $\mathcal{M} = s_0, s_1, s_2, \ldots$ which is isomorphic to the natural numbers, $\mathcal{N}$, and where each state, $s_i, 0 \leq i$, consists of the propositions that are true in it at the i-th moment of time.

Axiomatic system for $PLTL$ following [**?**]:

$(K_\Box)$	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
$(K_\bigcirc)$	$\vdash \bigcirc(p \to q) \to (\bigcirc p \to \bigcirc q)$
$(FUNC)$	$\vdash \bigcirc\neg p \leftrightarrow \neg \bigcirc p$
$(FP_\Box)$	$\vdash \Box p \leftrightarrow (\bigcirc p \wedge \bigcirc\Box p)$
$(GFP_\Box)$	$\vdash (q \wedge \Box(q \to (p \wedge \bigcirc q) \to \Box p$
$(FP_\mathcal{U})$	$\vdash p\mathcal{U}\, q \leftrightarrow (q \vee (p \wedge \bigcirc(p\mathcal{U}\, q)))$
$(LFP_\mathcal{U})$	$\vdash \Box((q \vee (p \wedge \bigcirc r)) \to r) \to (p\mathcal{U}\, q \to r)$

PLTL is used in the specification and verification of the reactive and concurrent programs [8] [7, pp. 381-386] and real-time programs [2, 3] (especially properties such as *safety*, *liveness*, and *fairness*); in temporal representation and reasoning [1, 6, 10].

Socratic Proofs

The Socratic transformations framework offers a formal representation of the idea of solving logical problems of entailment and derivability by pure questioning. This is the method of so called Socratic transformation of an initial question into consecutive questions without making any use of answers to the questions just transformed [9, 11]. Socratic transformations may be either successful or unsuccessful. Informally, a successful transformation, called a Socratic proof, ends with a question of a specified final form, which can be answered, from a semantic point of view, in only one rational way.

From a proof-theoretical point of view successful Socratic transformation, that is, a Socratic proof ends with an affirmative answer to the initial question concerning the derivability (or entailment) of a formula from a set of formulae. The rules governing Socratic transformations have only questions as premises and conclusions, and thus they form the core of an erotetic calculi (from Greek 'erotema' – 'question' [12]).

In a Socratic transformation, one transforms a question Q_1 into another question, Q_2, both written in a dedicated formalism L. A list of erotetic rules governs relevant transformations of questions of L. A question has a general form $?(A \vdash_L B)$ and is interpreted as: 'Is it the case that B is L-derivable from A?'.

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Normalisation for Negative Free Logic with Definite Descriptions

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Keywords: free logic, definite descriptions, natural deduction, proof theory.

Normalisation theorems establish that certain detours, to use Gentzen's phrase, in proofs in systems of natural deduction are unnecessary and can be removed by reduction procedures. This is familiar for intuitionist logic from Prawitz's work. The present talk focuses on issues arising from adapting Prawitz's methods to negative free logic with the ι operator for definite descriptions. The system to be considered was formulated by Tennant:

$$(\wedge I)\ \frac{A \qquad B}{A \wedge B} \qquad (\wedge E)\ \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B} \qquad (\rightarrow I)\ \frac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \rightarrow B}\,i$$

$$(\rightarrow E)\ \frac{A \rightarrow B \qquad A}{B}$$

$$(\vee I)\ \frac{A}{A \vee B} \quad \frac{B}{A \vee B} \qquad (\vee E)\ \frac{A \vee B \quad \begin{matrix}[A]^i \\ \Pi \\ C\end{matrix} \quad \begin{matrix}[B]^j \\ \Sigma \\ C\end{matrix}}{C}\,i,j \qquad (\bot E)\ \frac{\bot}{B}$$

$$(\forall I)\ \frac{\begin{matrix}[\exists! a]^i \\ \Pi \\ A^x_a\end{matrix}}{\forall x A}\,i \qquad (\forall E)\ \frac{\forall x A \quad \exists! t}{A^x_t} \qquad (\exists I)\ \frac{A^x_t \quad \exists! t}{\exists x A} \qquad (\exists E)$$

$$\frac{\exists x A \qquad \begin{matrix}[A^x_a]^i [\exists! a]^j \\ \Pi \\ C\end{matrix}}{C}\,i,j$$

where in $(\forall I)$, a does not occur in $\forall x A$ nor in any undischarged assumptions of Π except those in the assumption class of $\exists! a$; and where in $(\exists E)$, a is not free in C, nor in $\exists x A$, nor in any undischarged assumptions of Π except those in the assumption classes of A^x_a and $\exists! a$.

$$(=I^n)\ \frac{\exists! t}{t = t} \qquad (=E)\ \frac{t_1 = t_2 \quad A^x_{t_1}}{A^x_{t_2}} \qquad (AD)\ \frac{Rt_1 \ldots t_n}{\exists! t_i}$$

where R is an n-place predicate letter (but not $\exists!$) or identity and $1 \leq i \leq n$.

$$(\iota I)\ \frac{\exists! t \qquad \begin{matrix}[a{=}t]^i \\ \Xi \\ F^x_a\end{matrix} \qquad \begin{matrix}[F^x_a]^j [\exists! a]^k \\ \Pi \\ a{=}t\end{matrix}}{\iota x F = t}\, i, j, k$$

where a does not occur in $\iota x F$, nor in t, nor in any undischarged assumptions except those in the assumption classes of $a = t$ in Ξ or of F^x_a and $\exists! a$ in Π.

$$(\iota E_1)\ \frac{\iota x F = t \quad u = t}{F^x_u} \qquad (\iota E_2)\ \frac{\iota F x = t \quad F^x_u \quad \exists! u}{u = t} \qquad (\iota E_3)\ \frac{\iota F x = t}{\exists! t}$$

The issues I shall focus on are the following:

(a) Free logic requires alternative reduction procedures for the quantifiers, and they require an alternative measure of the complexity of $\forall x A$ and $\exists x A$, due to the presence of $\exists!$ in the rules.

(b) $(= E)$ can be restricted to prime conclusions, atomic formulas that do not contain ι terms.

(c) In negative free logic there are new kinds of detours to be considered. We can apply $(= I^n)$ and (AD) alternately: this is clearly superfluous. More interesting are sequences of applications of $(= E)$ that conclude the premise of (AD) or the minor premises of which are all of the form $\exists! t$. These are also superfluous, and there is a philosophical point to avoiding them: they make unnecessary existence assumptions and if ι terms are involved, they may also introduce concepts unncessary to deriving the conclusion of the deduction.

(d) The reduction procedures for the rules for the definite description operator given by Tennant pose the problem that ι terms of unknown complexity may be replaced for parameters and maximal formulas of unknown degree may be introduced into the deduction.

The last problem is the most interesting one. It is solved by modifying the rules for identity following a suggestion of Indrzejczak's. When $(= I^n)$ is replaced by

$$(= I^{nG})\ \frac{\exists! t \qquad \begin{matrix}[a{=}t]^i \\ \Pi \\ C\end{matrix}}{C}\, i$$

(ιI), (ιE_1) and (ιE_2) can be restricted to that only one ι term occurs in their applications, i.e. were it is displayed in the rules. The modification requires a change in the definition of the new maximal formulas and segments of point (c), but nothing essential.

In the normalisation proof, maximal segments of point (c) are treated separately, and are easy to remove from deductions. Once that is done, normalisation can proceed by an induction over the complexity of deductions similar to the one Prawitz uses for intuitionist logic.

A system of classical negative free logic with definite description arises from using the above introduction and elimination rules for $\rightarrow$, $\forall$, $=$, ι, (AD) and $(\bot E_C)$ instead of $(\bot E)$:

$$(\bot E_C)\ \frac{\begin{array}{c}[\neg A]^i\\ \Pi\\ \bot\end{array}}{A}\, i$$

As conclusions of $(\bot E_C)$ can only be restricted to atomic, but not prime, conclusions, new reduction procedures for conclusions of $(\bot E_C)$ that are the major premises of elimination rules for ι are required. To prove normalisation I adapt method of Andou's developed for normalisation for classical logic with $\vee$ and $\exists$.

Decision procedure for PLTL via the method of Socratic proofs

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Keywords: PLTL, decision procedure, the method of Socratic proofs, Inferential Erotetic Logic.

Temporal logic is used, int.al., to model purely logical aspects of reasoning about events in time, whereas erotetic logic is used, int.al., to model purely logical aspects of reasoning with questions involved. Many great logicians worked in both these branches of philosophical logic (some being considered as the founding fathers or at least pioneers in both of them—like Arthur Prior or Nuel Belnap), which makes at least one good reason to have the two—the temporal and the erotetic—in one framework.

The method of Socratic proofs is a proof method grounded in the logic of questions called Inferential Erotetic Logic [7,9,12]. The method has been adjusted to a number of various logics [3,5,6,8,10,11] and it can be viewed as a formal tool of analysis of cognitive processes which we would describe as inquiring into a problem, problem-solving, problem reformulating, dividing problems into subproblems or even answering questions by pure questioning [7,9].

The aims of our presentation are twofold. First, we will describe the erotetic calculus for Propositional Linear-Time Temporal Logic (PLTL) from purely proof-theoretical perspective. Technically, the calculus is a variant of sequent-calculus formulation of analytic tableaux, using ideas coming from natural deduction [1,2] and resolution systems [4] to deal with some specific problems of PLTL (like loops). We will present a decision procedure for PLTL and discuss its implementation in Python.

In the second part of the presentation we shall focus on the erotetic aspect of the method; we will discuss its potential in the modelling of reasoning involving questions.

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Cut-elimination for non-Fregean logic WB and some of its consequences

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Keywords: Non-Fregean logics, Boolean non-Fregean logic WB, cut-elimination and decidability.

Non-Fregean logics were proposed by Roman Suszko as the opposition to the idea, attributed to Frege, that sentences denote their own truth values. A critique of this view has been raised in Wittgenstein's *Tractatus*, which became a philosophical base for developing new logical systems by Suszko. Following Wittgenstein and Wolniewicz [14], he disagreed with the Fregean view on denotation arguing that identity of the logical value of two sentences does not entail identity of their denotations [12, 13].

Suszko provided syntactic and algebraic characterization of several non-Fregean systems; the weakest non-Fregean logic introduced by him, SCI (*Sentential Calculus with Identity*), was obtained from Classical Propositional Calculus (hereafter, CPC) by extending the language with the identity connective '$\equiv$', and by adding, as axioms, formulas of the following forms:

($\equiv_1$) $\phi \equiv \phi$

($\equiv_2$) $(\phi \equiv \chi) \to (\neg\phi \equiv \neg\chi)$

($\equiv_3$) $(\phi \equiv \chi) \to (\phi \leftrightarrow \chi)$

($\equiv_4$) $((\phi \equiv \psi) \wedge (\chi \equiv \omega)) \to ((\phi \otimes \chi) \equiv (\psi \otimes \omega))$

where $\otimes \in \{\wedge, \vee, \to, \leftrightarrow, \equiv\}$. Modus ponens (MP) is the only rule of inference. Identity characterized by SCI is very strong; every thesis of this logic with '$\equiv$' as the main connective follows under the first axiom scheme. Formulas of this form are called *trivial equations*.

The distinction between a truth value and a denotation of a sentence allows for a much more refined analysis of its content. However, the fact that only trivial equations are theorems of SCI makes the analysis drastically limited. Suszko himself found three theories, extending SCI, philosophically important. In this paper we focus on one of them, WB (W stands for 'Wittgenstein' and B – for 'Boolean'). In WB, the axiomatic basis of SCI is enriched with *Boolean axioms*, that is, formulas of the forms:

$(\equiv_5)$ $((\phi \wedge \chi) \vee \psi) \equiv ((\chi \vee \psi) \wedge (\phi \vee \psi))$
$(\equiv_6)$ $((\phi \vee \chi) \wedge \psi) \equiv ((\chi \wedge \psi) \vee (\phi \wedge \psi))$
$(\equiv_7)$ $(\phi \vee \bot) \equiv \phi$
$(\equiv_8)$ $(\phi \wedge \top) \equiv \phi$
$(\equiv_9)$ $(\phi \to \chi) \equiv (\neg\phi \vee \chi)$
$(\equiv_{10})$ $(\phi \leftrightarrow \chi) \equiv ((\phi \to \chi) \wedge (\chi \to \phi))$

Semantically, WB is interpreted in Boolean algebras.

Decidability of SCI was settled in [1], and sequent calculi were proposed already in the seventies [8, 10, 11]; later on a large number of proof systems and proof procedures for SCI has been developed [2–7, 9, 16, 17]. In the case of WB, for a long time the axiomatic account was the only available proof-theoretic description of the logic. Although WB seems relatively simple, its decidability has not been proved so far.

In [15] sequent calculi for three extensions of SCI, called WB, WT and WH, were developed. To the best of our knowledge, [15] introduced the first sequent systems for these logics. However, the author of [15] presented arguments that in the sequent calculus for WB the rule of cut is not eliminable. In our talk we will present a new variant of sequent calculus for WB, $\mathsf{SC}_{\mathsf{WB}}$, and we will demonstrate the proof of cut elimination in $\mathsf{SC}_{\mathsf{WB}}$. Using derived rules of $\mathsf{SC}_{\mathsf{WB}}$ we will define a decision procedure for WB.

The rules of $\mathsf{SC}_{\mathsf{WB}}$ are sound in the original algebraic semantics, but we also define an extension of a truth-valuation semantics for WB and use it in the proof of correctness of the procedure. Finally, we use $\mathsf{SC}_{\mathsf{WB}}$ as a point of departure for a definition of Kripke-style semantics for WB. It is well-known that the stronger extensions of SCI—WT and WH—correspond to modal logics S4 and S5, it is not settled, however, what is the modal content of WB. Our Kripke semantics opens up a possibility to adress this issue.

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Intuitionistic Natural Deduction with a General Collapse Rule and Its Propositions-as-Types Interpretation

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Keywords: natural deduction, proof-theoretic semantics, Curry-Howard correspondence, propositions-as-types principle.

We introduce a natural deduction system with a general collapse rule for intuitionistic propositional logic. The general collapse rule is a generalization of the collapse rule that replaces the falsity rule also known as the ex falso quodlibet rule. We provide a computational variant of the system in the style of the Curry-Howard correspondence and show that it is strongly normalizing.

The general collapse rule is a generalization of the rule (COL):

$$\frac{A \vee \bot}{A}\ \textsc{col}$$

introduced in [1] as an alternative for the falsity rule also known as the ex falso quodlibet rule (EFQ). With (COL), (EFQ) becomes a derived rule, and thus its justification can be reduced to the justification of (COL). The justification provided for (COL) was partly philosophical and partly technical.

Informally, (COL) expresses the idea that a choice between A and absurdity $\bot$ is not a proper choice and that it can be reduced to just A. So, it essentially works as a choice simplification rule. Thus, in contrast to (EFQ)'s explosion principle ("everything follows from absurdity"), it rather embodies *the implosion principle* ("a choice involving absurdity can be simplified"), which, however, still leads to explosion under certain conditions.

Formally, (COL) is treated as an elimination-like rule for disjunction that is both locally sound and complete (in the sense of [2]), unless in the presence of absurdity. If that is the case, it loses its local soundness. More specifically, when the premise $A \vee \bot$ of (COL) is constructed from $\bot$, the (COL) becomes too strong as an elimination rule, i.e., it will allow us to derive from $A \vee \bot$ more than the corresponding introduction rule for it permits, which is how the explosion principle is recaptured. In other words, the explosion principle is treated as a form of local unsoundness of the collapse rule in the presence of $\bot$.

Some of the technical questions were, however, left open in [1]. Most importantly, whether a natural deduction system with (COL) instead of (EFQ) retains normalization. In this talk, we will address this question by introducing a natural deduction system with the collapse rule for intuitionistic propositional logic IPC called $\mathbf{N}^{c}$ that enjoys the strong normalization property. More specifically, we will show it for a typed variant $\mathbf{N}_t^{c}$ utilizing the Curry-Howard correspondence. For the proof itself, we will be relying on Tait's reducibility method ([3]).

However, rather than using a natural deduction system with the original collapse rule (COL), as mentioned earlier, we introduce its generalized version together with its typed variant equipped with a selector function (see below). We will call it the (C) rule:

$$\frac{A \vee \bot \qquad \begin{matrix}[A]^i \\ C\end{matrix}}{C}\ \mathrm{c}_i$$

The main motivation for adopting this generalized version instead of (COL) is that it simplifies the search for the appropriate selector function for its typed variant in the style of the Curry-Howard correspondence, which will be needed for the normalization proof. However, as will be shown, the rules (COL) and (C) are logically and computationally interdefinable. More specifically, we can derive (COL) from (C) and vice versa. And, analogously, we can define the (COL)'s selector via the selector of (C) and vice versa. Thus, we can also claim that normalization holds for a natural deduction system with (COL) as well.

The typed variant of the (C) will be as follows:

$$\frac{c : A \vee \bot \qquad \begin{matrix}[x : A]^i \\ d(x) : C\end{matrix}}{\mathsf{implode}(c, x.d) : C}\ \mathrm{c}_i$$

with the following computation rule for the implode selector:

$$\mathsf{implode}(c, x.d) : C \Longrightarrow \begin{cases} d(a) : C & \text{if } c \Longrightarrow \mathsf{i}(a) : A \vee \bot \text{ with } a : A \\ \varnothing & \text{otherwise} \end{cases}$$

In short, $\mathsf{implode}(\mathsf{i}(a), x.d)$ reduces to $d(a)$, while $\mathsf{implode}(\mathsf{j}(b), x.d)$ is a further irreducible term, i.e., a value. So, when implode encounters absurdity it behaves similarly to the selector for the typed ex falso quodlibet rule (EFQ) which also has no computation rule. Actually, the selector implode can be used to define the selectors collapse, explode, and ds for the typed variants of the original collapse rule, ex falso quodlibet rule, and disjunctive syllogism, respectively.

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Intuitionistic Logic of Paradox: Introducing a Paraconsistent Intuitionistic Logic

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Keywords: paraconsistent logic, paraconsistency, intuitionistic logic, intuitionism, logic of paradox, Kripke semantics, dialetheism, three-valued logic, law of excluded middle, double negation elimination, recapture.

In the philosophical literature on the foundations of logic, various principles of classical logic have been questioned and weaker systems have been proposed. In paraconsistent logics, the law of explosion fails: a theory may contain a sentence together with its negation without exploding into triviality [1]. In intuitionistic logic, the law of excluded middle fails: it may be that a sentence does not hold, but neither does its negation. As noted by Tedder and Shapiro, these two non-classical systems correspond to largely disjoint areas of research [6].

We aim to contribute to bridging this gap by introducing a paraconsistent intuitionistic logic, in which both laws fail. Building on the bachelor's thesis under supervision of Robert Passmann [5], we take as a starting point Priest's paraconsistent logic *propositional logic of paradox* (**LP**) [4] and combine it with the Kripke semantics for intuitionistic logic [2]. The result is a natural extension of both logics, which we call *intuitionistic logic of paradox* (**ILP**). We introduce two versions of this logic; one that obeys modus ponens, and one that does not (remaining faithful to **LP**).

Whenever a logic is weakened, this raises the question of *recapture*: can the original logic be recovered from its weakening by adding certain assumptions? We raise this question for **ILP** with respect to **LP** and intuitionistic logic. By its very definition, **ILP** allows for a primitive kind of recapture of both these logics: restricting the consequence relation to a certain class of models allows us to recover the consequence relation of **LP** or intuitionistic logic.

However, such recapture is not necessarily expressible within the logic. We might hope for something stronger – in particular, we ask the question: does **ILP** stand to **LP** as intuitionistic logic stands to classical logic? Classical logic can be recovered from intuitionistic logic by adding the law of excluded middle; do we obtain **LP** when we strengthen **ILP** by adding this law? The question turns out not to be straightforward, and its answer depends on the formulation of the law of excluded middle – intuitionistically equivalent formulations are no longer equivalent in **ILP** – and on the version of **ILP** that is used.

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Algebraic Completeness and Semantic Cut-Elimination of a Multimodal Formulation of Morrrill's Categorial Logic

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Keywords: Multimodal Lambek Calculus, Substructural Modalities, Subexponentials.

In this paper we present a new algebraic formulation of Morrill's CatLog system, a Type-Logical Calculus which accounts for grammaticality in Natural Languages (NL) in terms of theoremhood. CatLog has been studied in depth in several papers from a proof-theoretic point of view. Crucially the CatLog's algebraic account of non-linearity linguistic phenomena in this article is done with no S4 modalities, only with Moortgat style substructural modalities and (modal) structural rules. We prove Cut-elimination and completeness with respect a well-defined algebraic semantics. Moreover, the proof of Cut-elimination is algebraic, not proof-theoretic. In the paper we correct the proposed type-assigments of relative pronouns (in several papers over the years, at least from 2015), avoiding thus undergeneration of very basic relative clause phenomena in NL. Finally, our new algebraic framework is used to improve a publeshed Soft Linear Logic account of Catlog.