Definite Descriptions in First-Order Temporal Setting

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- We will present a sound and complete **tableau system** for $\mathsf{FOHL}^{P,F}_{\iota,\lambda}$.
- Finally, using the tableau system we will show that $\mathsf{FOHL}^{P,F}_{\iota,\lambda}$ enjoys Craig's interpolation property.

UNDERLYING THEORY OF DEFINITE DESCRIPTIONS

Russell’s eliminativist approach

Our starting point is **Russell’s approach to DDs** characterised by the following formula:

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Example

$$\psi(y) := \boxed{A(y) \to A(y)} \qquad \varphi(x) := \boxed{B(x) \land \neg B(x)}$$

$$[\boxed{A(\iota x(\boxed{B(x) \land \neg B(x)})) \to A(\iota x(\boxed{B(x) \land \neg B(x)}))}] \leftrightarrow$$

$$\exists y[\forall x((B(x) \land \neg B(x)) \leftrightarrow x = y) \land (A(y) \to A(y))]$$

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(a) restricted predication of DDs to predicate abstracts of the form $\lambda x \psi$,

(b) modified (R) accordingly:

$$(\lambda x \psi)(\iota y\, \varphi) \leftrightarrow \exists x (\forall y (\varphi \leftrightarrow y = x) \wedge \psi). \qquad (R_\lambda)$$

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$\boxed{d_1}$: ‘the oldest daughter of Anne’

$$\iota x(D(x,a) \land \forall y(D(y,a) \land y \neq x \to O(x,y)))$$

$\boxed{d_2}$: the richest woman in the family

$$\iota x(\forall y(F(y,a) \land W(y) \land y \neq x \to R(x,y)))$$

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Then (**S**) can be formalised as:

$$(\lambda x(\exists y(B(y) \land M(x,y)) \land (\lambda y(x = y))(\boxed{d_2})))(\boxed{d_1})$$

FIRST-ORDER
HYBRID TEMPORAL LOGIC

Hybrid logic

Hybrid logic

Hybrid logic (HL) is an extension of standard modal logic. The language of HL is given by the following grammar:

$$\varphi := p \mid i \mid x \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid @_i \varphi \mid \downarrow_x \varphi$$

where p is a propositional variable, i is a **nominal** and x is a **state variable**.

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A model of HL is an extension of Kripke model for standard modal logic such that **each nominal is assigned a singleton set** by the valuation function.

Moreover, we define an **assignment** as a function assigning to each state variable a world from the universe of a model.

Hybrid logic

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\mathscr{M}, \mathscr{A}, w \models @_{\boldsymbol{i}} \varphi \quad & \text{iff} \quad \mathscr{M}, \mathscr{A}, v \models \varphi \quad \text{and} \quad \mathscr{V}(i) = \{v\}
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\mathscr{M}, \mathscr{A}, w \models \downarrow_{\boldsymbol{x}} \varphi & \text{iff} & \mathscr{M}, \mathscr{A}[\boldsymbol{x} \mapsto w], w \models \varphi,
\end{array}
$$

where $\mathscr{A}[\boldsymbol{x} \mapsto w]$ is an assignment identical to $\mathscr{A}$ except that it assigns the world w to $\boldsymbol{x}$.

First-order hybrid temporal logic

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$$\begin{aligned}\mathrm{FOR} \ni \varphi ::= {} & \bot \mid P(\mathscr{s}_1, \ldots, \mathscr{s}_n) \mid \mathscr{s}_1 = \mathscr{s}_2 \mid \boldsymbol{t} \mid \neg\varphi \mid \\ & \varphi \wedge \varphi \mid \mathrm{F}\varphi \mid \mathrm{P}\varphi \mid \exists x\, \varphi \mid \lambda x\, \varphi(x)(t) \mid @_{\boldsymbol{t}}\, \varphi \mid {\downarrow_{\boldsymbol{x}}}\, \varphi\text{[**]},\end{aligned}$$

where $\mathscr{s}_1, \ldots, \mathscr{s}_n$ are terms which are not DDs.

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[**] Hybrid "counterparts" of object terms use the same symbols as the latter, but are written in bold. And so, e.g., x is an object variable, whereas $\boldsymbol{x}$ is a state variable, i is an object constant, whereas $\boldsymbol{i}$ is a nominal, etc.

FOHL$_{\iota,\lambda}^{\mathsf{P,F}}$ semantics

Definition

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- $\mathscr{I}$ is an **interpretation function**, where:
  - for each n-ary predicate P and $w \in \mathscr{W}$, $\mathscr{I}(P, w) \subseteq \mathscr{D}^n$
  - for each object constant i, $\mathscr{I}(i) \in \mathscr{D}$
  - for each nominal $\boldsymbol{i}$, $\mathscr{I}(\boldsymbol{i}) \in \mathscr{W}$.

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An **$\text{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}$ assignment** $\mathscr{A}$ is a function such that:

- for each object variable x, $\mathscr{A}(x) \in \mathscr{D}$
- for each state variable $\boldsymbol{x}$, $\mathscr{A}(\boldsymbol{x}) \in \mathscr{W}$.

$\mathrm{FOHL}_{\iota,\lambda}^{P,F}$ satisfaction conditions

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$\mathscr{M}, w, \mathscr{A} \models \lambda x \varphi(\imath y \psi)$ iff there exists $o \in \mathscr{D}$ such that $\mathscr{M}, w, \mathscr{A}[y \mapsto o] \models \psi$ and $\mathscr{M}, w, \mathscr{A}[x \mapsto o] \models \varphi$, and for any $o' \in \mathscr{D}$, if $\mathscr{M}, w, \mathscr{A}[y \mapsto o'] \models \psi$, then $o' = o$

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$\mathscr{M}, w, \mathscr{A} \models \iota \boldsymbol{x}\, \varphi$	iff	$\mathscr{M}, w, \mathscr{A}[\boldsymbol{x} \mapsto w] \models \varphi$ and for any $v \in \mathscr{W}$, if $\mathscr{M}, v, \mathscr{A}[\boldsymbol{x} \mapsto v] \models \varphi$, then $v = w$.

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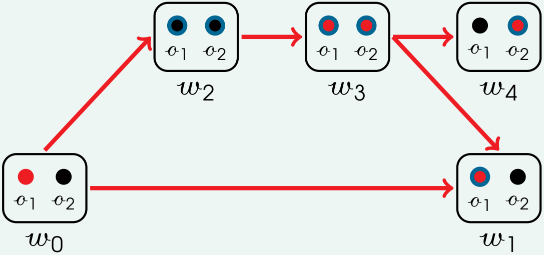
Example

Let's consider the following $\mathrm{FOHL}_{\iota,\lambda}^{\mathrm{P,F}}$ model:



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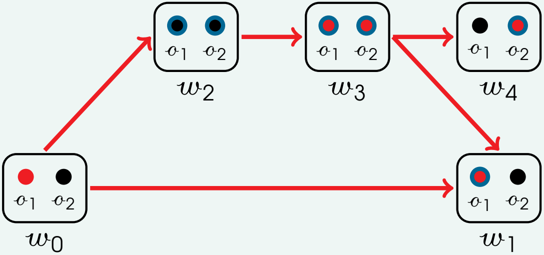

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○ object o is bald ($B(o)$)

and an assigment $\mathscr{A}$ which maps **all object variables to o_1 and all state variables to w_0**.

Example (cont'd)

We can make the following observations:

- the formula $\lambda x B(x)(\iota y K(y))$ ("The present king of France is bald.") is satisfied at time instances w_1 and w_4 by, respectively, o_1 and o_2

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- at w_0 the formula $\lambda x \neg B(x)(\iota y K(y))$ ("The present king of France is not bald.") is true, whereas at w_2 and w_3 it is false
- at w_0 and w_3 the formula $\mathbf{F}\lambda x B(x)(\iota y K(y))$ ("At some point in the future the then-present king of France will be bald") holds, whereas at w_2 it fails to hold.

TABLEAU SYSTEM

Tableu rules

We propose an **internalised tableau system** for $\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}$, abbreviated as $\mathsf{TC}_{\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$.

Below, we only present rules specific for $\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}$. For the remaining rules see, e.g., the work of Bolander and Blackburn [3].

Quantifier rules

$$(\exists)\ \frac{@_{\boldsymbol{i}} \exists x\, \varphi}{@_{\boldsymbol{i}} \varphi[x/\mathscr{a}]}{}^{*} \qquad\qquad (\neg\exists)\ \frac{\neg @_{\boldsymbol{i}} \exists x\, \varphi}{\neg @_{\boldsymbol{i}} \varphi[x/\mathscr{b}]}{}^{**}$$

Nominal rules

$$(\mathrm{eq})\ \frac{@_{\boldsymbol{i}} \mathscr{b} = \mathscr{b}'}{\mathscr{b} = \mathscr{b}'}{}^{**} \qquad\qquad (\neg\mathrm{eq})\ \frac{\neg @_{\boldsymbol{i}} \mathscr{b} = \mathscr{b}'}{\mathscr{b} \neq \mathscr{b}'}{}^{**}$$

---

* $\mathscr{a}$ is a fresh parameter (free object variable).

** each of $\mathscr{b}, \mathscr{b}'$ is a parameter or an object constant occurring on the branch.

Tableau rules

ι object rules

$$(\iota_1^{\circ})\ \frac{@_{i}\lambda x\psi(\iota y\,\varphi)}{\begin{array}{c}@_{i}\,\varphi[y/\mathscr{a}]\\ @_{i}\,\psi[x/\mathscr{a}]\end{array}}{}^{*}\qquad\qquad (\iota_2^{\circ})\ \frac{\begin{array}{c}@_{i}\lambda x\psi(\iota y\,\varphi)\\ @_{i}\,\varphi[y/\mathscr{b}]\\ @_{i}\,\varphi[y/\mathscr{b}']\end{array}}{\mathscr{b}=\mathscr{b}'}{}^{**}$$

$$(\neg\iota^{\circ})\ \frac{\neg @_{i}\lambda x\psi(\iota y\,\varphi)}{\neg @_{i}\,\psi[x/\mathscr{b}]\ \mid\ \neg @_{i}\,\varphi[y/\mathscr{b}]\ \mid\ \begin{array}{c}@_{i}\,\varphi[y/\mathscr{a}]\\ \mathscr{a}\neq\mathscr{b}\end{array}}{}^{*,**}$$

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Tableau rules

ι temporal rules

$$(\iota_1^\dagger)\ \frac{@_{\boldsymbol{i}}\, \iota \boldsymbol{x}\, \varphi}{@_{\boldsymbol{i}}\, \varphi[\boldsymbol{x}/\boldsymbol{i}]} \qquad (\iota_2^\dagger)\ \frac{\begin{array}{c}@_{\boldsymbol{i}}\, \iota \boldsymbol{x}\, \varphi \\ @_{\boldsymbol{i}'}\, \varphi[\boldsymbol{x}/\boldsymbol{i}']\end{array}}{@_{\boldsymbol{i}}\, \boldsymbol{i}'}$$

$$(\neg\iota^\dagger)\ \frac{\neg @_{\boldsymbol{i}}\, \iota \boldsymbol{x}\, \varphi}{\neg @_{\boldsymbol{i}}\, \varphi[\boldsymbol{x}/\boldsymbol{i}] \mid \begin{array}{c}@_{\boldsymbol{j}}\, \varphi[\boldsymbol{x}/\boldsymbol{j}] \\ \neg @_{\boldsymbol{i}}\, \boldsymbol{j}\end{array}}^{*}$$

$$(@\iota^\dagger)\ \frac{@_{\boldsymbol{i}} @_{\iota \boldsymbol{x}\, \varphi} \psi}{\begin{array}{c}@_{\boldsymbol{j}}\, \iota \boldsymbol{x}\, \varphi \\ @_{\boldsymbol{j}}\, \psi\end{array}}^{*} \qquad (\neg @\iota^\dagger)\ \frac{\neg @_{\boldsymbol{i}} @_{\iota \boldsymbol{x}\, \varphi} \psi}{\neg @_{\boldsymbol{i}'}\, \iota \boldsymbol{x}\, \varphi \mid \neg @_{\boldsymbol{i}'}\, \psi}^{**}$$

---

* $\boldsymbol{j}$ is a fresh nominal.

** $\boldsymbol{i}'$ is a nominal occurring on the branch.

Tableau rules

λ rules

$$(\lambda)\ \frac{@_i \lambda x \psi(\mathscr{b})}{@_i \psi[x/\mathscr{b}]} \qquad (\neg\lambda)\ \frac{\neg @_i \lambda x \psi(\mathscr{b})}{\neg @_i \psi[x/\mathscr{b}]}$$

Other rules

$$(\text{ref})\ \frac{}{\mathscr{b} = \mathscr{b}}{}^{*} \qquad (\text{RR})\ \frac{\begin{matrix}\varphi(\mathscr{b}) \\ \mathscr{b} = \mathscr{b}'\end{matrix}}{\varphi[\mathscr{b}//\mathscr{b}']}{}^{**} \qquad (\text{NED})\ \frac{}{\mathscr{a} = \mathscr{a}}{}^{***}$$

---

[*] $\mathscr{b}$ is a parameter or an object constant occurring on the branch.

[**] $\varphi[\mathscr{b}//\mathscr{b}']$ is a formula φ in which some occurrences of $\mathscr{b}$ were replaced by occurrences of $\mathscr{b}'$.

[***] $\mathscr{a}$ is a fresh parameter. The rule can be applied at most once in case:

- we make a non-empty domain assumption
- no other rules are applicable and there are neither parameters nor object constants on the branch.

Example

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Hence they moved to London at the year of Brexit.

It may be formalised in a simplified form (avoiding details not relevant for the validity of this example) in the following way:

$$@_{\iota \boldsymbol{x} W(t,j)} M(t, j,), \quad @_{\iota \boldsymbol{x} W(t,j)} \iota \boldsymbol{y} B \quad \vdash \quad @_{\iota \boldsymbol{y} B} M(t, j,).$$

Example (cont'd)

Below we present the proof of correctness of the above argument:

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1. $@_{\boldsymbol{i}_1} @_{\iota \boldsymbol{x} W(t,j)} M(t, j,)$
2. $@_{\boldsymbol{i}_1} @_{\iota \boldsymbol{x} W(t,j)} \imath \boldsymbol{y} B$
3. $\neg @_{\boldsymbol{i}_1} @_{\iota \boldsymbol{y} B} M(t, j,)$

Example (cont'd)

Below we present the proof of correctness of the above argument:

$$1.\quad @_{\boldsymbol{i}_1} @_{\iota \boldsymbol{x} W(t,j)} M(t, j,)$$

$$2.\ @_{\boldsymbol{i}_1} @_{\iota \boldsymbol{x} W(t,j)} \imath \boldsymbol{y} B$$

$$3.\ \neg @_{\boldsymbol{i}_1} @_{\iota \boldsymbol{y} B} M(t, j,)$$

$$\big| \ (@\iota^{\dagger}): 2$$

$$4.\ @_{\boldsymbol{i}_2} \iota \boldsymbol{x} W(t, j)$$

$$5.\ @_{\boldsymbol{i}_2} \iota \boldsymbol{y} B$$

Example (cont'd)

Below we present the proof of correctness of the above argument:

1. $@_{\boldsymbol{i}_1} @_{\iota \boldsymbol{x} W(t,j)} M(t, j,)$

2. $@_{\boldsymbol{i}_1} @_{\iota \boldsymbol{x} W(t,j)} ℩ \boldsymbol{y} B$

3. $\neg @_{\boldsymbol{i}_1} @_{\iota \boldsymbol{y} B} M(t, j,)$

$\mid$ $(@\iota^{\dagger})$: 2

4. $@_{\boldsymbol{i}_2} \iota \boldsymbol{x} W(t, j)$

5. $@_{\boldsymbol{i}_2} \iota \boldsymbol{y} B$

$\mid$ $(@\iota^{\dagger})$: 1

6. $@_{\boldsymbol{i}_3} ℩ \boldsymbol{x} W(t, j)$

7. $@_{\boldsymbol{i}_3} M(t, j,)$

Example (cont'd)

Below we present the proof of correctness of the above argument:

$$1.\quad @_{\boldsymbol{i}_1} @_{\iota \boldsymbol{x} W(t,\jmath)} M(t, \jmath,)$$

$$2.\quad @_{\boldsymbol{i}_1} @_{\iota \boldsymbol{x} W(t,\jmath)} \imath \boldsymbol{y} B$$

$$3.\quad \neg @_{\boldsymbol{i}_1} @_{\iota \boldsymbol{y} B} M(t, \jmath,)$$

$\mid$ $(@\iota^{\dagger})$: 2

$$4.\quad @_{\boldsymbol{i}_2} \iota \boldsymbol{x} W(t, \jmath)$$

$$5.\quad @_{\boldsymbol{i}_2} \iota \boldsymbol{y} B$$

$\mid$ $(@\iota^{\dagger})$: 1

$$6.\quad @_{\boldsymbol{i}_3} \imath \boldsymbol{x} W(t, \jmath)$$

$$7.\quad @_{\boldsymbol{i}_3} M(t, \jmath,)$$

$$\ldots$$

Example (cont'd)

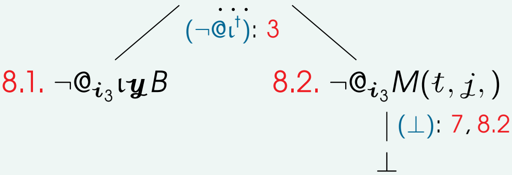
$\dots$

$(\neg@\iota^{\dagger})$: 3

8.1. $\neg@_{\boldsymbol{i}_3}\iota\boldsymbol{y}B$

8.2. $\neg@_{\boldsymbol{i}_3}M(t, j,)$

Example (cont'd)



$\dots$

$(\neg @\iota^{\dagger})$: 3

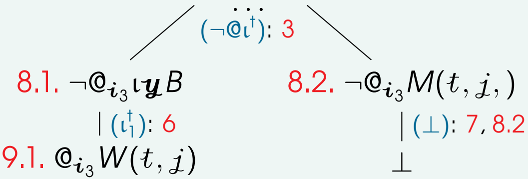
8.1. $\neg @_{\boldsymbol{i}_3} \iota \boldsymbol{y} B$

8.2. $\neg @_{\boldsymbol{i}_3} M(t, j,)$

$(\bot)$: 7, 8.2

$\bot$

Example (cont'd)



$\dots$

$(\neg @\iota^{\dagger})$: 3

8.1. $\neg @_{\boldsymbol{i}_3} \iota \boldsymbol{y} B$

$(\iota_1^{\dagger})$: 6

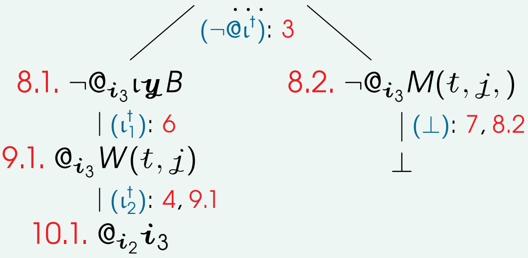
9.1. $@_{\boldsymbol{i}_3} W(t, j)$

8.2. $\neg @_{\boldsymbol{i}_3} M(t, j,)$

$(\bot)$: 7, 8.2

$\bot$

Example (cont'd)



$\dots$

$(\neg @\iota^{\dagger})$: 3

8.1. $\neg @_{\boldsymbol{i}_3} \iota \boldsymbol{y} B$

$|$ $(\iota^{\dagger}_1)$: 6

9.1. $@_{\boldsymbol{i}_3} W(t, j)$

$|$ $(\iota^{\dagger}_2)$: 4, 9.1

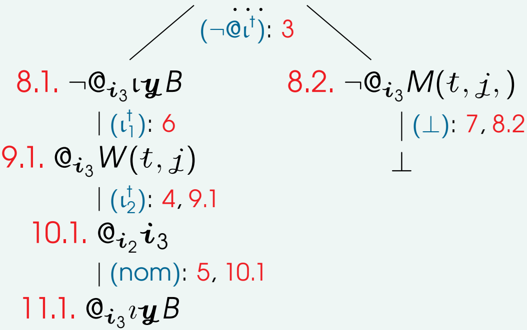
10.1. $@_{\boldsymbol{i}_2} \boldsymbol{i}_3$

8.2. $\neg @_{\boldsymbol{i}_3} M(t, j,)$

$|$ $(\bot)$: 7, 8.2

$\bot$

Example (cont'd)



…

$(\neg @\iota^{\dagger})$: 3

8.1. $\neg @_{\boldsymbol{i}_3} \iota \boldsymbol{y} B$

$|$ $(\iota^{\dagger}_1)$: 6

9.1. $@_{\boldsymbol{i}_3} W(t, j)$

$|$ $(\iota^{\dagger}_2)$: 4, 9.1

10.1. $@_{\boldsymbol{i}_2} \boldsymbol{i}_3$

$|$ (nom): 5, 10.1

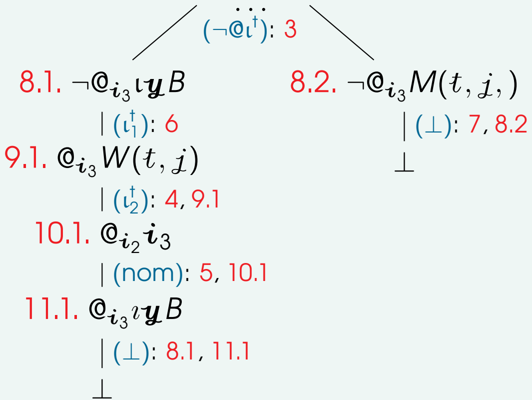
11.1. $@_{\boldsymbol{i}_3} \imath \boldsymbol{y} B$

8.2. $\neg @_{\boldsymbol{i}_3} M(t, j,)$

$|$ $(\perp)$: 7, 8.2

$\perp$

Example (cont'd)



$\dots$

$(\neg @\iota^{\dagger})$: 3

8.1. $\neg @_{\boldsymbol{i}_3} \iota \boldsymbol{y} B$

$\mid (\iota^{\dagger}_1)$: 6

9.1. $@_{\boldsymbol{i}_3} W(t, j)$

$\mid (\iota^{\dagger}_2)$: 4, 9.1

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$\mid$ (nom): 5, 10.1

11.1. $@_{\boldsymbol{i}_3} \text{℩} \boldsymbol{y} B$

$\mid (\bot)$: 8.1, 11.1

$\bot$

8.2. $\neg @_{\boldsymbol{i}_3} M(t, j,)$

$\mid (\bot)$: 7, 8.2

$\bot$

Soundness and completeness

Theorem

The tableau system $\mathsf{TC}_{\mathsf{FOHL}^{P,F}_{\iota,\lambda}}$ is **sound and complete** with respect to the semantics of $\mathsf{FOHL}^{P,F}_{\iota,\lambda}$.

Proof: [6]. ■

Interpolation

Preliminaries

To prove interpolation for $\mathrm{FOHL}^{P,F}_{\iota,\lambda}$ we need several auxiliary results.

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Proposition

The rule: (cut) $\dfrac{}{\varphi \mid \neg\varphi}$ is admissible in the tableau system $\mathsf{TC}_{\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$.

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Proposition

The rule: (cut) $\dfrac{}{\varphi \mid \neg\varphi}$ is admissible in the tableau system $\mathsf{TC}_{\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$.

Let $\mathbf{TC}'_{\mathbf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$ be the tableau system $\mathsf{TC}_{\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$ with the following rule transformations:

$$(\iota_2^{\circ}) \quad \rightsquigarrow \quad (\iota_2^{\circ\prime})\ \frac{@_{\boldsymbol{i}}\lambda x \psi(\iota y\, \varphi)}{\neg @_{\boldsymbol{i}}\, \varphi[y/\mathscr{b}] \mid \neg @_{\boldsymbol{i}}\, \varphi[y/\mathscr{b}'] \mid \mathscr{b} = \mathscr{b}'}$$

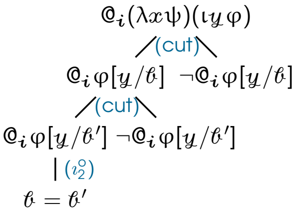
$$(\iota_2^{\dagger}) \quad \rightsquigarrow \quad (\iota_2^{\dagger\prime})\ \frac{@_{\boldsymbol{i}}\, \iota \boldsymbol{x}\, \varphi}{\neg @_{\boldsymbol{i}'}\, \varphi[\boldsymbol{x}/\boldsymbol{i}'] \mid @_{\boldsymbol{i}}\, \boldsymbol{i}'}$$

Proposition

The calculi $\mathsf{TC}_{\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$ and $\mathsf{TC}'_{\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$ are equivalent.

Proof: We prove the proposition by showing that (ι_2°) and $(\iota_2^{\circ\prime})$ as well as $(\iota_2^{\dagger})$ and $(\iota_2^{\dagger\prime})$ are interderivable.

For the former pair consider the following derivation trees:



$@_i(\lambda x\psi)(\iota y\,\varphi)$

(cut)

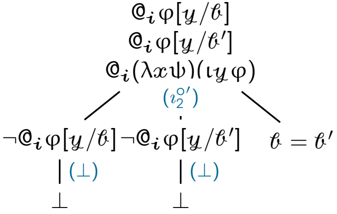
$@_i\varphi[y/\mathscr{b}]$ $\neg @_i\varphi[y/\mathscr{b}]$

(cut)

$@_i\varphi[y/\mathscr{b}']$ $\neg @_i\varphi[y/\mathscr{b}']$

(ι_2°)

$\mathscr{b} = \mathscr{b}'$



$@_i\varphi[y/\mathscr{b}]$

$@_i\varphi[y/\mathscr{b}']$

$@_i(\lambda x\psi)(\iota y\,\varphi)$

$(\iota_2^{\circ\prime})$

$\neg @_i\varphi[y/\mathscr{b}]$ $\neg @_i\varphi[y/\mathscr{b}']$ $\mathscr{b} = \mathscr{b}'$

$(\bot)$ $(\bot)$

$\bot$ $\bot$

The interderivability of the other pair of rules can be shown analogously. ■

Finding interpolant

Suppose that we have an implication $\varphi \to \psi$ that is valid in $\mathsf{FOHL}^{\mathsf{P},\mathsf{F}}_{\iota,\lambda}$.

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1. From the closed $\mathsf{TC}'_{\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$ tableau for $\neg @_{\boldsymbol{i}}(\varphi \to \psi)$, where $\boldsymbol{i}$ does not occur in $\varphi \to \psi$, we delete the root and replace $@_{\boldsymbol{i}}\varphi$ with $\mathtt{L}@_{\boldsymbol{i}}\varphi$ and $@_{\boldsymbol{i}}\neg\psi$ with $\mathtt{R}@_{\boldsymbol{i}}\neg\psi$.

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2. Going **downwards** in the tableau we **assign** $\mathtt{L}$ **and** $\mathtt{R}$ to each formula so that whenever the premise of a rule is signed with $\mathtt{X}$, for $\mathtt{X} \in \{\mathtt{L}, \mathtt{R}\}$, then the conclusions of the rule are signed with $\mathtt{X}$.

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3. Going **upwards** in the tableau we **assign an interpolant** to a formula, each time based on the principle formulated for the applied rule [4, 2].

We build our interpolant-finding technique upon the methods provided by Fitting [4] and Blackburn and Marx [2].

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Example

$(\mathtt{L}\neg\iota^{\circ})$ If χ_1 is an interpolant for $\Gamma \cup \{\mathtt{L}\ \neg @_{i}\psi[y/b]\}$, χ_2 is an interpolant for $\Gamma \cup \{\mathtt{L}\ \neg @_{i}\varphi[y/b]\}$ and χ_3 is an interpolant for $\Gamma \cup \{\mathtt{L}\ @_{i}\varphi[y/a], \mathtt{L}\ a \neq b\}$, then $\forall x(\chi_1 \vee \chi_2 \vee \chi_3)[b/x]$ is an interpolant for $\Gamma \cup \{\mathtt{L}\ \neg @_{i}\lambda x\psi(\iota y\,\varphi)\}$.

$(\mathtt{R}\neg\iota^{\circ})$ If χ_1 is an interpolant for $\Gamma \cup \{\mathtt{R}\ \neg @_{i}\psi[y/b]\}$, χ_2 is an interpolant for $\Gamma \cup \{\mathtt{R}\ \neg @_{i}\varphi[y/b]\}$ and χ_3 is an interpolant for $\Gamma \cup \{\mathtt{R}\ @_{i}\varphi[y/a], \mathtt{R}\ a \neq b\}$, then $\exists x(\chi_1 \wedge \chi_2 \wedge \chi_3)[b/x]$ is an interpolant for $\Gamma \cup \{\mathtt{R}\ \neg @_{i}\lambda x\psi(\iota y\,\varphi)\}$.

Theorem

If $\varphi \rightarrow \psi$ is $\mathrm{FOHL}_{\iota,\lambda}^{P,F}$-valid, then there exists a formula θ such that $\varphi \rightarrow \theta$ and $\theta \rightarrow \psi$ are also $\mathrm{FOHL}_{\iota,\lambda}^{P,F}$-valid and all non-logical expressions occurring in θ occur in both φ and ψ.

Proof: [6]. ■

Beth definability

An immediate consequence of Craig's interpolation theorem for $\mathrm{FOHL}_{\iota,\lambda}^{P,F}$ is Beth's definability theorem:

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An immediate consequence of Craig's interpolation theorem for $\mathsf{FOHL}^{P,F}_{\iota,\lambda}$ is Beth's definability theorem:

Theorem

Let Th be a $\mathsf{FOHL}^{P,F}_{\iota,\lambda}$-theory and let ξ be a non-logical expression (that is, a predicate or constant) occurring in Th. Then ξ is implicitly definable under Th if and only if it is explicitly definable under Th.

Constant elimination

The Beth definability property allows us to check if, for a given theory Th and constant c occurring in Th, **c is dispensable under Th**.

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The following two conditions are equivalent:

1. $\mathrm{Th} \cup \boxed{\mathrm{Th}'} \models c = c'$
2. $\exists\psi\left[\mathrm{Th} \models \forall x, \overline{y}(x = c \leftrightarrow \boxed{\psi(x, \overline{y})})\right]$

$\boxed{\mathrm{Th}'}$: the theory Th with c' instead of c, where c' is fresh

$\boxed{\psi}$: an $\mathrm{FOHL}^{\mathrm{P,F}}_{\iota,\lambda}$-formula where c does not occur

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$\boxed{\psi}$: an $\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}$-formula where c does not occur

To decide whether ψ exists, it thus suffices to check with $\mathsf{TC}_{\mathsf{FOHL}^{\mathsf{P,F}}_{\iota,\lambda}}$ if the formula

$$\bigwedge(\mathsf{Th} \cup \mathsf{Th}') \wedge c \neq c'$$

is satisfiable [1].

Example

Consider a theory Th which provides characteristics of two individuals: **Charles** and **Dana**.

1. Charles is a politician.
2. Dana is a politician.
3. No one else is a politician.

Formally:

$$\text{Th} = \{P(c), \quad P(d), \quad \forall x(P(x) \rightarrow (x = c \vee x = d))\}$$

(+ all the formulas logically entailed by the above ones.)

Example (cont'd)

It is easy to check that d **is implicitly definable in Th**:

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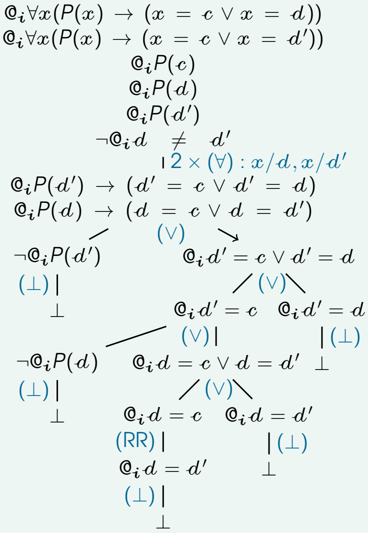

$$@_i \forall x (P(x) \to (x = c \lor x = d))$$
$$@_i \forall x (P(x) \to (x = c \lor x = d'))$$
$$@_i P(c)$$
$$@_i P(d)$$
$$@_i P(d')$$
$$\neg @_i d \neq d'$$

$2 \times (\forall) : x/d, x/d'$

$$@_i P(d') \to (d' = c \lor d' = d)$$
$$@_i P(d) \to (d = c \lor d = d')$$

$(\lor)$

$\neg @_i P(d')$ $\quad$ $@_i d' = c \lor d' = d$

$(\bot)$ $\quad$ $(\lor)$

$\bot$ $\quad$ $@_i d' = c$ $\quad$ $@_i d' = d$

$(\lor)$ $\quad$ $(\bot)$

$\neg @_i P(d)$ $\quad$ $@_i d = c \lor d = d'$ $\quad$ $\bot$

$(\bot)$ $\quad$ $(\lor)$

$\bot$ $\quad$ $@_i d = c$ $\quad$ $@_i d = d'$

(RR) $\quad$ $(\bot)$

$@_i d = d'$ $\quad$ $\bot$

$(\bot)$

$\bot$

Since Th does not specify whether Charles and Dana are the same person, the explicit definition of d is the following:

$$\psi(x) := P(x) \wedge \big(x \neq c \vee \neg\exists y(y \neq x \wedge P(y))\big),$$

saying that either Dana is a politician distinct from Charles or the only politician that exists. Thus, d can be replaced with

$$\iota x(\psi(x))$$

in every syntactically allowed context.

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1. If we **remove** $P(c)$ from Th, d **will still be** explicitly definable under Th with $\psi(x)$.
2. If we **remove** $P(d)$ from Th, d **will no longer be** explicitly definable under Th.

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Thank You.

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