Decidability in First-Order Modal Logic with Non-Rigid Constants and Definite Descriptions

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ExtenDD Seminar
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Overview

1. Introduction
2. Preliminaries
3. Results
   - Related Formalisms and Reductions
   - Quasimodels and Weak Quasimodels
   - Temporal Logics
4. Conclusion

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Introduction and Motivations

Non-Rigid Designators and Counting Features (NRDC)

First-order modal logics (FOMLs) extended with:

- **Non-rigid designators**: non-rigid constants and definite descriptions
- **Counting** (non-trivial): equality or counting quantifiers

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Philosophical Applications

- **Referential opacity** in modal contexts (with failure of substitutivity for equality)

  e.g., ‘the number of planets is necessarily greater than 7’ vs. ‘8 is necessarily greater than 7’

- **Descriptivist** vs. **direct reference** theories of proper names

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KR Applications

- **Epistemic** and **temporal logics**: individual symbols denoting distinct objects in alternative conceivable scenarios or over time
- **Free logics**, **description logics**, **hybrid logics**, ...

Background and Challenges

The Bad

Modal extensions of decidable FO fragments are typically **undecidable**, e.g.:

- Monadic fragment of FO **decidable**
- Monadic fragment of FOMLs $\mathbf{K}_n$ and $\mathbf{S5}_n$, with $n \geq 1$, **undecidable**

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Monodic fragments: modalities applied only to formulas with $\leq$ **1 free variable**

- often preserve decidability of underlying FO fragments...
- ...but rely on the absence of NRDC features!

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- ...but rely on the absence of NRDC features!

The Ugly

Mostly **negative results** on computational behaviour of fragments with NRDC features

- from product modal logics & fragments of FO modal/temporal logics with counting

Our Contribution

Investigation of **decidability** and **complexity boundaries**
for **monodic fragments** with **NRDC features**

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- **definite descriptions**

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Dimensions

- **Fragments**: **monodic** with **FO restrictions** (1-var., 2-var. + counting, guarded)

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- **Decision problems**: **validity** and **global consequence**
  - Global consequence **not reducible** to validity in general, but...
  - ...**reducible** on frames w/: single equivalence relation; transitive closure; or time flows

Overview of Results

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$	$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$
S5	$=$	coNExp	coNExp	2Exp	coNExp	coNExp	2Exp
S5$_n$, $n \geq 2$	$=$	coNExp	coNExp	2Exp	undecidable		
K$_n$	$=$	coNExp	coNExp	2Exp	undecidable		
	$\subseteq$	PSpace	coNExp	2Exp	?		
K$_{*n}$, **LTL**$^{(\Diamond)}$	$=$	Σ^1_1					
	$\subseteq$	undecidable					
K*f*$_{*n}$, **LTL*f***$^{(\Diamond)}$	$=$	undecidable					
	$\subseteq$	decidable, Ackermann-hard					

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$Q^{=}ML_{\iota}$, the First-Order Modal Language with NRDC

Definition (Terms and Formulas)

$Q^{=}ML_{\iota}$ **terms** and **formulas** defined by mutual induction:

$$\tau ::= x \mid c \mid \iota x.\varphi$$
$$\varphi ::= P(\tau_1, \ldots, \tau_m) \mid \tau_1 = \tau_2 \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \exists x\, \varphi \mid \Diamond_a \varphi$$

- **variables** $x \in \text{Var}$, **constants** $c \in \text{Con}$, and **predicates** $P \in \text{Pred}$ (m-ary)
- finite set of **modalities** $a \in A$
- **definite descriptions** $\iota x.\varphi$, read as "the x such that φ"

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Notation

- $\varphi_1 \vee \varphi_2 := \neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \rightarrow \varphi_2 := \neg\varphi_1 \vee \varphi_2$, $\varphi_1 \leftrightarrow \varphi_2 := (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$
- $\forall x\, \varphi := \neg\exists x\, \neg\varphi$
- $\Box_a \varphi := \neg\Diamond_a \neg\varphi$
- Γ finite set of sentences (no free variables)

Semantics with Partial Interpretations

Definition (Partial Interpretation with Expanding Domains)

$\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$ where:

- $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ **frame** with worlds W ($\neq \emptyset$) and accessibility relations R_a
- Δ function assigning **domain** Δ_w ($\neq \emptyset$) to each $w \in W$ s.t. $\Delta_w \subseteq \Delta_v$ when wR_av
- $\cdot$ function mapping each $w \in W$ to **partial FO interpretation** $\mathfrak{M}(w)$ with:
  - $P^{\mathfrak{M}(w)} \subseteq \Delta_w^m$, for each m-ary predicate $P \in \mathrm{Pred}$ (total on predicates)
  - $c^{\mathfrak{M}(w)} \in \Delta_w$, for **some constant** symbols $c \in \mathrm{Con}$ (**partial** on constants)

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Definition (Designation, Total Interpretations, Constant Domains, Assignments)

- c **designates at** w: $c^{\mathfrak{M}(w)}$ is defined
- **total interpretation**: all constants designate at all worlds
- **constant domains**: $\Delta_w = \Delta_v$ for all $w, v \in W$
- **assignment at** w: function $\mathfrak{a}$ from Var to Δ_w
- x-**variant** of $\mathfrak{a}$ at w: assignment $\mathfrak{a}'$ at w that can differ from $\mathfrak{a}$ only on x

Term Interpretation and Satisfaction

Definition (Value of Terms)

$$\tau^{\mathfrak{M}(w),\mathfrak{a}} = \begin{cases} \mathfrak{a}(x), & \text{if } \tau = x \in \mathsf{Var} \\ c^{\mathfrak{M}(w)}, & \text{if } \tau = c \in \mathsf{Con} \text{ and } c^{\mathfrak{M}(w)} \textbf{ defined} \\ \mathfrak{a}'(x), & \text{if } \tau = \iota x.\varphi \text{ and } \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi \text{ for } \textbf{exactly one } x\text{-variant } \mathfrak{a}' \text{ of } \mathfrak{a} \\ \textbf{undefined}, & \text{otherwise} \end{cases}$$

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Definition (Satisfaction Relation)

- $\mathfrak{M}, w \models^{\mathfrak{a}} P(\tau_1, \ldots, \tau_m)$ iff all $\tau_i^{\mathfrak{M}(w),\mathfrak{a}}$ **defined** and $(\tau_1^{\mathfrak{M}(w),\mathfrak{a}}, \ldots, \tau_m^{\mathfrak{M}(w),\mathfrak{a}}) \in P^{\mathfrak{M}(w)}$
- $\mathfrak{M}, w \models^{\mathfrak{a}} \tau_1 = \tau_2$ iff both $\tau_i^{\mathfrak{M}(w),\mathfrak{a}}$ **defined** and $\tau_1^{\mathfrak{M}(w),\mathfrak{a}} = \tau_2^{\mathfrak{M}(w),\mathfrak{a}}$
- $\mathfrak{M}, w \models^{\mathfrak{a}} \exists x\, \varphi$ iff there exists x-variant $\mathfrak{a}'$ with $\mathfrak{M}, w \models^{\mathfrak{a}'} \varphi$
- $\mathfrak{M}, w \models^{\mathfrak{a}} \Diamond_a \varphi$ iff there exists $v \in W$ such that $wR_a v$ and $\mathfrak{M}, v \models^{\mathfrak{a}} \varphi$

Truth, Validity, and Global Consequence

Definition (Truth, Satisfaction)

- φ **true in** $\mathfrak{M}$, $\mathfrak{M} \models \varphi$: φ satisfied under every assignment at every world of $\mathfrak{M}$
- φ **satisfied in** $\mathfrak{M}$: φ satisfied under some assignment at some world of $\mathfrak{M}$
- Γ **true in** $\mathfrak{M}$, $\mathfrak{M} \models \Gamma$: every sentence in Γ is true in $\mathfrak{M}$

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Definition ($\mathbf{K}_n$ and $\mathbf{S5}_n$ Frames)

- $\mathbf{K}_n$: class of all frames with n accessibility relations
- $\mathbf{S5}_n$: class of frames with n equivalence relations; $\mathbf{S5} := \mathbf{S5_1}$

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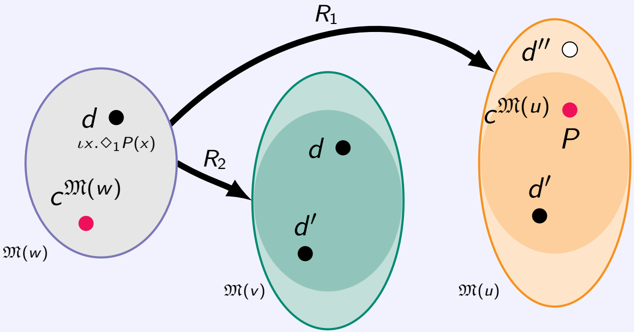
Definition (Validity, Satisfiability, Global Consequence)

$\mathcal{C}$ class of frames. Formula φ

- $\mathcal{C}$-**valid**: φ true in every interpretation $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathcal{C}$
- $\mathcal{C}$-**satisfiable**: φ satisfied in some interpretation $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathcal{C}$
- **global** $\mathcal{C}$**-consequence of theory** Γ: φ true in any interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that $\mathfrak{M} \models \Gamma$

Examples

Example (Partial Interpretation with Non-Rigid Designators)



$$\exists x(x \neq c \wedge \Diamond_1(x = c))$$

unsatisfiable if **constants** are **rigid**

Examples

Example (Vulcan and Venus)

- "It is conceivable *that* Vulcan is the planet orbiting between Sun and Mercury":

$$\Diamond(\text{vulcan} = \iota z.\text{OrbitsBetween}(z, \text{sun}, \text{mercury}))$$

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- "Even though such a planet does not exist":

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  i.e., neither 'vulcan' nor '$\iota z.\mathrm{OrbitsBetween}(z, \mathrm{sun}, \mathrm{mercury})$' designate in this world

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- “It is known *of* the planet orbiting between Mercury and Earth that it is Venus”:

$$\exists x(x = \iota z.\mathrm{OrbitsBetween}(z, \mathrm{mercury}, \mathrm{earth}) \land \Box(x = \mathrm{venus}))$$

  in **S5** frames, this implies that ‘venus’ is rigid

Monodic Fragments

Definition (Monodic Fragment)

Set $Q^{=}_{\boxed{1}}ML_{\iota}$ of **monodic** formulas:

- every subformula of the form $\Diamond_{a}\psi$ has **at most one free variable**.

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Minimal Sub-Fragment

- $\mathrm{Q}^{1=}\mathrm{ML}_{\iota}$: **One-variable** fragment with $\leq$ 1-ary predicates (and equality)

Maximal Sub-Fragments

- $\mathrm{C}^{2}_{\boxed{1}}\mathrm{ML}_{\iota}$: **Two-variable** fragment with **counting** quantifiers and $\leq$ 2-ary predicates
- $\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_{\iota}$: **Guarded** fragment with equality

One-Variable Fragment $Q^{1=}ML_\iota$

Definition (One-Variable Fragment $Q^{1=}ML_\iota$)

$Q^{1=}ML_\iota$ **terms** and **formulas** built with:

- **one variable** only
- predicates of **arity at most one** (plus equality)
- constants and definite descriptions

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Remarks

- Underpinned by FO^1 with equality and constants
- All formulas trivially monodic
- Variants extensively studied as product modal logics

Two-Variable Fragment with Counting $C^2_{\boxed{1}}ML_\iota$

Definition (Two-Variable Fragment with Counting $C^2_{\boxed{1}}ML_\iota$)

$C^2_{\boxed{1}}ML_\iota$ **terms** and **formulas** built with:

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- **counting quantifiers** $\exists^{\geq k}x$, $k \geq 0$
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Definition (Counting Quantifier)

$$\mathfrak{M}, w \models^{\mathfrak{a}} \exists^{\geq k} x\, \varphi \quad \text{iff} \quad \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi \text{ for at least } k \text{ distinct } x\text{-variants } \mathfrak{a}'$$

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Example (Number of Planets)

$$\varphi_1 = \Diamond \exists^{\leq 9} x\, \mathsf{Planet}(x), \qquad \varphi_2 = \exists^{\leq 9} x\, \Diamond \mathsf{Planet}(x)$$

- Constant and expanding domains: $\varphi_1 \not\Rightarrow \varphi_2$
- Constant but not expanding domains: $\varphi_2 \Rightarrow \varphi_1$

Guarded Fragment $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_{\iota}$

Definition (Guarded Fragment $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_{\iota}$)

$\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_{\iota}$ **terms** and **formulas** built with:

- **guarded quantifiers** $\exists x_1 \cdots \exists x_k(\alpha \wedge \varphi)$, where α atom with all free variables of φ
- constants and **closed** definite descriptions $\iota x.\chi(x)$, $\chi(x)$ with ≤ 1 free variable x

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Remark

Closed definite descriptions necessary for decidability even without modalities

- $\forall x F(x, \iota y.F(x, y))$ ensures F is a **function**
- guarded fragment with **functionality** is **undecidable**

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$\exists x\, \varphi(x)$ can still be expressed

- equivalent to $\exists x\, ((x = x) \land \varphi(x))$, with $x = x$ as its guard.

Decision Problems

Definition (Validity and Global Consequence Decision Problems)

For fragment $\mathcal{L}$ and frame class $\mathcal{C}$

- $\mathcal{C}$-**validity in** $\mathcal{L}$: Is φ valid on all interpretations based on $\mathcal{C}$-frames?
- **global** $\mathcal{C}$-**consequence in** $\mathcal{L}$: Is φ true in all interpretations based on $\mathcal{C}$-frames that make theory Γ true?

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Problem Qualifiers

- **total $\mathcal{C}$-validity**: restriction to total interpretations
- with **constant** domains, with **expanding** domains

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Naming Convention

$$\text{Fragment } \mathcal{L} \text{ with } \begin{cases} \text{subscript } \iota\text{: both constants and definite descriptions} \\ \text{subscript } c\text{: only constants} \\ \text{no subscript: neither constants nor definite descriptions} \end{cases}$$

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Related Formalisms and Reductions

Quasimodels and Weak Quasimodels

Temporal Logics

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Overview of Techniques

Preliminary Observations

- Show correspondence between one-variable fragment with **difference**/**elsewhere** quantifier $\exists^{\neq} x$ and our one-variable fragment with **non-rigid constants**
  - Lift results from propositional case: "**difference** $\equiv$ **nominals** + **universal** modality"

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  - Lift results from propositional case: "**difference** $\equiv$ **nominals** + **universal** modality"
- Simplify the landscape
  - Normalise and **eliminate definite descriptions**
  - **Reduce partial** to **total** interpretations (and **viceversa**)
  - **Reduce expanding** to **constant** domains

Overview of Techniques

Main Ideas for Decidability

- Adapt **quasimodel** machinery for NRDC features
  - Use **multisets** of types and runs to handle counting

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  - 2-variable with counting: introduce weak pre-quasimodels + (in)equalities encoding constraints in Presburger Arithmetic with infinity
- **Expanding domains** simplify some cases:
  - validity/global consequence in fragments $\begin{cases} \text{with transitive closure \& no infinite chain} \\ \text{on finite temporal frames} \end{cases}$
  - $\mathbf{K}_n$-validity in one-variable fragment

Overview of Techniques

Main Ideas for Undecidability

- For global consequence on $\mathbf{K}_n$ ($n \geq 1$) and $\mathbf{S5}_n$ ($n \geq 2$) frames, reduce:
  - **undecidable products** to one-variable fragment with **elsewhere quantifier** $\exists^{\neq} x$
  - the latter to our one-variable fragment with **non-rigid constants**

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  - the latter to our one-variable fragment with **non-rigid constants**
- For validity/global consequence with **transitive closure** & on **temporal** (infinite with constant/expanding domains, or finite with constant domains) frames, reduce
  - **undecidable one-variable FOTL with counting** to our one-variable fragments on temporal (infinite or finite, resp.) frames
  - the latter to one-variable fragments with **transitive closure** (with or without infinite chains, resp.)

Overview

1 Introduction

2 Preliminaries

3 Results
   - Related Formalisms and Reductions
   - Quasimodels and Weak Quasimodels
   - Temporal Logics

4 Conclusion

One-Variable Fragment with Elsewhere $Q^{1\neq}ML$

Definition (One-Variable Fragment with Elsewhere $Q^{1\neq}ML$)

$Q^{1\neq}ML$ **formulas**:

$$\varphi ::= \; P(x) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \; \exists x\, \varphi \mid \exists^{\neq} x\, \varphi \mid \Diamond_a \varphi$$

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Definition (Elsewhere Quantifier)

$\mathfrak{M}, w \models^{\mathfrak{a}} \exists^{\neq} x\, \varphi$ iff $\mathfrak{M}, w \models^{\mathfrak{a}'} \varphi$, for some x-variant $\mathfrak{a}'$ of $\mathfrak{a}$ at w **different from** $\mathfrak{a}$

Reductions Between Non-Rigid Constants and Elsewhere Quantifier

Theorem

For any class of frames $\mathcal{C}$, with both constant and expanding domains

- *$\mathcal{C}$-validity in $\mathbf{Q^{1=}ML}_c$ and $\mathbf{Q^{1\neq}ML}$ are mutually polytime-reducible*
- *same applies to global $\mathcal{C}$-consequence*

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Proof (Idea).

- $\mathrm{Q^{1=}ML}_c$ to $\mathrm{Q^{1\neq}ML}$

$$x = c \quad \overset{\dagger}{\rightsquigarrow} \quad Q_c(x) \land \neg\exists^{\neq} x\, Q_c(x)$$

- $\mathrm{Q^{1\neq}ML}$ to $\mathrm{Q^{1=}ML}_c$ (global $\mathcal{C}$-consequence)

$$\Gamma \models \exists^{\neq} x\, \psi \quad \overset{\ddagger}{\rightsquigarrow} \quad \Gamma^{\ddagger} \cup \{\mathrm{singl}_\psi\} \models \exists x\, P_\psi(x) \land \big(x = c_\psi \rightarrow \exists x\, (\neg(x = c_\psi) \land P_\psi(x))\big)$$

where $\mathrm{singl}_\psi := \forall x\, \big(\psi(x) \rightarrow P_\psi(x)\big) \;\land\; \forall x\, \big(P_\psi(x) \rightarrow \psi(x) \land \psi(c_\psi)\big)$ □

Overview of Results for $\mathbf{K}_n$ and $\mathbf{S5}_n$ Global Consequence

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$	$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$
S5	$=$	coNExp	coNExp	2Exp	coNExp	coNExp	2Exp
$\mathbf{S5}_n$, $n \geq 2$	$=$	coNExp	coNExp	2Exp	undecidable		
$\mathbf{K}_n$	$=$	coNExp	coNExp	2Exp	undecidable		
	$\subseteq$	PSpace	coNExp	2Exp	?		
$\mathbf{K}_{*n}$, $\mathbf{LTL}^{(\Diamond)}$	$=$	Σ^1_1					
	$\subseteq$	undecidable					
$\mathbf{K}f_{*n}$, $\mathbf{LTL}f^{(\Diamond)}$	$=$	undecidable					
	$\subseteq$	decidable, Ackermann-hard					

First Undecidability Results

Theorem

For **constant** *domains:*

- **global $\mathbf{K}_n$-consequence** *with* $n \geq 1$ *in* $\mathrm{Q}^{1=}\mathrm{ML}_c$ *is* **undecidable**
- **global $\mathbf{S5}_n$-consequence** *with* $\boldsymbol{n \geq 2}$ *in* $\mathrm{Q}^{1=}\mathrm{ML}_c$ *is* **undecidable**

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Proof (Idea).

From undecidability of product modal logic $\mathbf{K}_u \times \mathbf{Diff}$, with $\mathbf{K}_u$ extending $\mathbf{K}$ with universal modality u, and **Diff** propositional modal logic of elsewhere

- Reduce from $\mathbf{K}_u \times \mathbf{Diff}$-validity to $\mathrm{Q}^{1\neq}\mathrm{ML}$ global $\mathbf{K}_n$-consequence
  - product world $\rightsquigarrow$ domain object at a world
  - $\Diamond_{\neq} \rightsquigarrow \exists^{\neq} x$
  - $\Diamond_u \rightsquigarrow$ global $\mathbf{K}_n$-consequence
- Then, use previous reduction from $\mathrm{Q}^{1\neq}\mathrm{ML}$ global $\mathbf{K}_n$-consequence to $\mathrm{Q}^{1=}\mathrm{ML}_c$
- Finally (for 2nd point), use known reduction from $\mathbf{K}$ to $\mathbf{S5}_2$ □

$\mathbf{S5}_1$ and $\mathbf{K}_n$ with Expanding Domains?

Exception: Global $\mathbf{S5}_1$-consequence

The theorem does **not** hold for **S5**, since global **S5**-consequence reduces to **S5**-validity:

$$\Gamma \models_{\mathbf{S5}} \varphi \quad \text{iff} \quad \Box \bigwedge \Gamma \rightarrow \varphi \text{ is } \mathbf{S5}\text{-valid}$$

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Open Problem: Global $\mathbf{K}_n$-Consequence with Expanding Domains

- Reduction from product modal logics works only for constant domains
- $\mathbf{S5}_n$ domains always constant (by symmetry of accessibility relation)
- Expanding domain case for global $\mathbf{K}_n$-consequence remains open

Simplifying the Landscape

Theorem

For $\mathcal{L} \in \{\mathsf{Q}^{1=}\mathsf{ML}_\iota, \mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota, \mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota, \mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_\iota\}$ *and any class of frames* $\mathcal{C}$*:*

1. $\mathcal{C}$*-validity in* $\mathcal{L}$ *is polytime-reducible to* $\mathcal{C}$*-validity in* $\mathcal{L}$ **w/out definite descriptions**
2. $\mathcal{C}$*-validity in* **partial** *and* **total** *interpretations are mutually polytime-reducible*
3. $\mathcal{C}$*-validity with* **expanding domains** *is polytime-reducible to* **constant-domains**

All hold also for global $\mathcal{C}$*-consequence in* $\mathcal{L}$

Simplifying the Landscape

Eliminating Definite Descriptions

Normalisation Procedure

- Replace definite descriptions $\iota x.\psi(x, \boldsymbol{y})$ with "atomic" ones: $\iota x.P_\psi(x, \boldsymbol{y})$
- Add "surrogates" for definite description "bodies": $\forall x \forall \boldsymbol{y}(P_\psi(x, \boldsymbol{y}) \leftrightarrow \psi(x, \boldsymbol{y}))$
- Iterate starting from innermost descriptions

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Elimination of Definite Descriptions

Replace atoms $\alpha(\iota x.Q(x, \boldsymbol{y}), \boldsymbol{\tau})$ with "**Russell's paraphrase**"

- in $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_\iota$

$$\exists x(\alpha(x, \boldsymbol{\tau}) \land Q(x, \boldsymbol{y}) \land \forall x'(Q(x', \boldsymbol{y}) \rightarrow x' = x))$$

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- in $\mathrm{C}^{2}_{\boxed{1}}\mathrm{ML}_\iota$
$$\exists x(\alpha(x, \boldsymbol{\tau}) \land Q(x, \boldsymbol{y})) \land \exists^{=1} x Q(x, \boldsymbol{y})$$
- in $\mathrm{Q}^{1=}\mathrm{ML}_\iota$ and $\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_\iota$, with $c_{\iota x.Q(x)}$ fresh constant symbol
$$\alpha(c_{\iota x.Q(x)}, \boldsymbol{\tau}) \land Q(c_{\iota x.Q(x)}) \land \forall x \left(Q(x) \to x = c_{\iota x.Q(x)}\right)$$

From Partial to Total

Introduce **propositional letter** p_c for each constant c to encode **whether** c **designates**

$$P(\tau_1, \ldots, \tau_m) \quad \rightsquigarrow \quad \bigwedge_{c_i \in \{\tau_1, \ldots, \tau_m\}} p_{c_i} \wedge P(\tau_1, \ldots, \tau_m)$$

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From Total to Partial

Add "**existence axioms**" to ensure that **each constant** c **designates**

$$\exists x (x = c)$$

Reduction from Expanding to Constant Domains

- By previous two points, in $\mathcal{L}$ *without* definite descriptions, reduce *total* $\mathcal{C}$-validity with expanding domains to *total* $\mathcal{C}$-validity with constant domain
- Use well-known reduction, with a semi-rigid (monotonically increasing) "**actuality predicate**" to encode expanding domains

Enforcing Infinite Branching

Example (Infinitely Branching Interpretations with Equality or Counting)

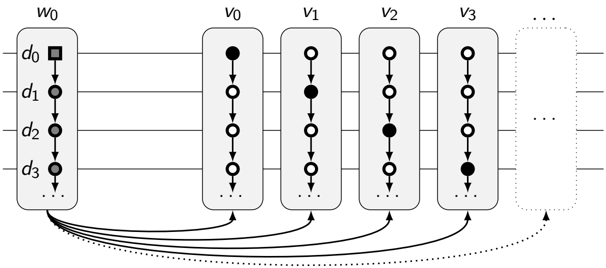
C^2-sentence φ_0 with only infinite models:

$$\forall x \exists^{=1} y\, P(x,y) \wedge \forall x \exists^{\leq 1} z\, P(z,x) \wedge \exists x \neg \exists z\, P(z,x)$$

$C^2_{\boxed{1}}ML_c$-sentence forcing infinite branching

$$\varphi = \varphi_0 \wedge \forall x\, \Diamond_a A(x) \wedge \Box_a \exists^{\leq 1} x\, A(x)$$

Each element in infinite P-chain requires separate a-successor with unique A-element



Overview

1. Introduction
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Quasimodels for $Q^{=}_{\boxed{1}}ML_c$

Main Ideas

- **Quasistates** describe worlds in interpretations

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Quasimodels for $Q^{=}_{\boxed{1}}ML_c$

Main Ideas

- **Quasistates** describe worlds in interpretations
- **Runs** correspond to domain elements
- Generalise basic quasimodels to handle **non-rigid constants** & **equality/counting**
  - use **multisets** of types and runs to take care of cardinalities

Types

Definition (Surrogates)

Surrogate $\overline{\varphi}$: replace modal subformulas of φ

- monodic $\Diamond_a\psi(x) \rightsquigarrow$ unary predicate $R_{\Diamond_a\psi}(x)$
- monodic $\Diamond_a\psi \rightsquigarrow$ propositional variable $p_{\Diamond_a\psi}$

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Definition (Type)

Type for $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence φ: subset $t \subseteq sub_x(\varphi)$, where for fresh variable x

$$sub_x(\varphi) = \{\psi\{x/y\}, \neg\psi\{x/y\} \mid \psi(y) \in sub(\varphi)\}$$

that is Boolean-saturated, i.e., for every sub-formula $\psi_1 \land \psi_2, \neg\psi \in sub_x(\varphi)$

$$\psi_1 \land \psi_2 \in t \text{ iff } \psi_1 \in t \text{ and } \psi_2 \in t; \qquad \neg\psi \in t \text{ iff } \psi \notin t$$

Surrogate type $\overline{t} := \{\overline{\psi} \mid \psi \in t\}$

Quasistates and Basic Structures

Definition (Multiset)

Set X equipped (& identified) with **multiplicity function** $\boldsymbol{X}(x) \in \mathbb{N} \cup \{\aleph_0\}$, for $x \in X$

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Definition (Quasistate Candidate)

Quasistate candidate for φ: non-empty multiset $\boldsymbol{n}$ of types for φ with multiplicity $\boldsymbol{n}(t)$

- $\boldsymbol{n}$ **realised** in FO structure $\mathfrak{B}$: $\boldsymbol{n}(t) = |\{b \in \mathfrak{B} \mid \mathfrak{B} \models \bar{t}[b]\}|$

Quasistate for φ: **realisable** quasistate candidate (i.e., realised by some $\mathfrak{B}$)

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Quasistate for φ: **realisable** quasistate candidate (i.e., realised by some $\mathfrak{B}$)

Definition (Basic Structure)

$$(\mathfrak{F}, \boldsymbol{q}), \text{ where } \begin{cases} \mathfrak{F} = (W, \{R_a\}_{a \in A}) \text{ frame} \\ \boldsymbol{q} \text{ function assigning quasistate } \boldsymbol{q}(w) \text{ to each } w \in W \end{cases}$$

Runs

Definition (Run)

Run ρ in $(\mathfrak{F}, \boldsymbol{q})$: map from worlds w in upward-closed $W' \subseteq W$ to types $\rho(w) \in \boldsymbol{q}(w)$:

- **(r-coh)** $\exists v \in W : wR_a v$ and $\psi \in \rho(v) \Rightarrow \Diamond_a \psi \in \rho(w)$
- **(r-sat)** $\Diamond_a \psi \in \rho(w) \Rightarrow \exists v \in W : wR_a v$ and $\psi \in \rho(v)$

Domain of ρ: $\mathrm{dom}\rho = W' \subseteq W$ (upward-closed); **full run**: $\mathrm{dom}\rho = W$

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Definition (Multiset of Runs)

$\mathfrak{R}$ **multiset of runs**:

- w-**slice** $\mathfrak{R}_w \subseteq \mathfrak{R}$, where $\mathfrak{R}_w(\rho) = \begin{cases} \mathfrak{R}(\rho), & \text{if } w \in \mathrm{dom}\rho, \\ 0, & \text{otherwise.} \end{cases}$
- (w, t)-**slice** $\mathfrak{R}_{w,t} \subseteq \mathfrak{R}$, where $\mathfrak{R}_{w,t}(\rho) = \begin{cases} \mathfrak{R}(\rho), & \text{if } w \in \mathrm{dom}\rho \text{ and } \rho(w) = t, \\ 0, & \text{otherwise.} \end{cases}$

Multiset $\mathfrak{R}$ of runs $\rightsquigarrow$ set of **indexed runs** $\hat{\mathfrak{R}} = \{(\rho, \ell) \in \mathfrak{R} \times \mathbb{N} \mid 0 \leq \ell < \mathfrak{R}(\rho)\}$

Quasimodels

Definition (Quasimodel)

- **(Expanding-domain) quasimodel** $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ for φ:
  - $(\mathfrak{F}, \boldsymbol{q})$ basic structure for φ
  - $\mathfrak{R}$ multiset of runs through $(\mathfrak{F}, \boldsymbol{q})$ such that

    (card) $\boldsymbol{q}(w, t) = |\mathfrak{R}_{w,t}|$ for all $w \in W$ and types t for φ
- **Constant-domain quasimodel**: $\mathfrak{R}$ consists of full runs
- $\mathfrak{Q}$ **satisfies** φ: $\varphi \in t$ for some $w_0 \in W$ and $t \in \boldsymbol{q}(w_0)$

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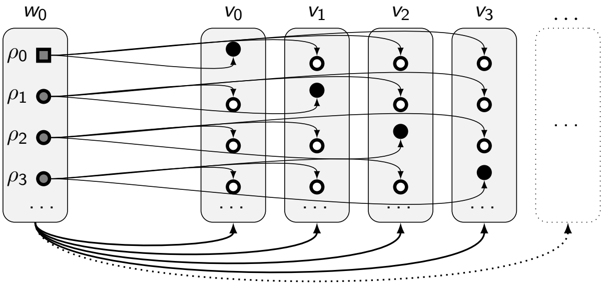

Quasimodels and Interpretations

Lemma (Quasimodel Lemma)

For both constant and expanding domains, $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$*-sentence* φ **satisfiable** *in interpretation based on frame* $\mathfrak{F}$ **iff** *there* **exists quasimodel satisfying** φ *based on* $\mathfrak{F}$

Quasimodels and Interpretations

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For both constant and expanding domains, $Q^{=}_{\boxed{1}}ML_c$*-sentence* φ **satisfiable** *in interpretation based on frame* $\mathfrak{F}$ **iff** *there* **exists quasimodel satisfying** φ *based on* $\mathfrak{F}$

Proof (Idea).

($\Rightarrow$) Given interpretation $\mathfrak{M}$ satisfying φ:

- types: $t^{\mathfrak{M}(w)}(d) := \{\psi \in sub_x(\varphi) \mid \mathfrak{M}, w \models \psi[d]\}$
- quasistate candidates: $\boldsymbol{q}(w, t) :=$ number of domain elements realizing t at w
- runs: $\rho_d(w) := t^{\mathfrak{M}(w)}(d)$, for $d \in \Delta_w$

($\Leftarrow$) Given quasimodel $\mathfrak{Q}$ satisfying φ:

- Construct interpretation with domains $\Delta_w = \hat{\mathfrak{R}}_w$
- Use bijections $f_w : \hat{\mathfrak{R}}_w \to$ domain of $\mathfrak{B}_w$ realising $\boldsymbol{q}(w)$ to define
  - $c^{\mathfrak{M}(w)} = f_w^{-1}(c^{\mathfrak{B}_w})$
  - $P^{\mathfrak{M}(w)} = f_w^{-1}(P^{\mathfrak{B}_w})$

□

Infinite Branching Again (Recall)

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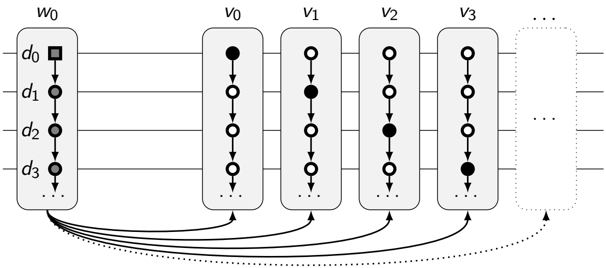

Infinite Branching Again

Infinite Branching in Quasimodels

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Infinite Branching Again

Infinite Branching in Quasimodels

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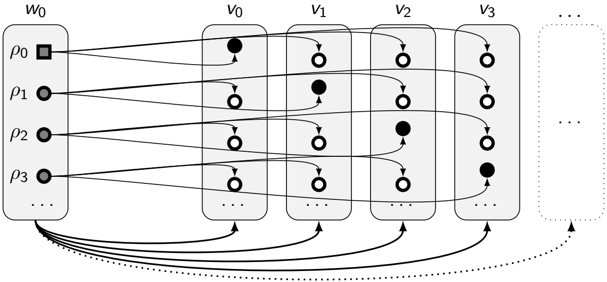
  - Infinite Branching Example $\rightsquigarrow$ infinite branching quasimodels
    - type "gray circle" has $\aleph_0$ infinite multiplicity in w_0
    - multiset of runs contains $\aleph_0$ runs (of multiplicity 1) through type "gray circle" at w_0
    - by quasimodel constraints, each of $\aleph_0$ runs then requires a **separate** ***a*-successor**



Infinite Branching Again

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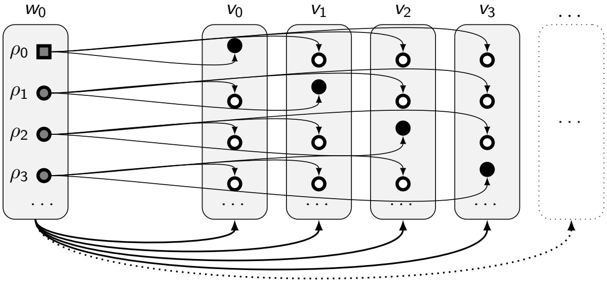

- Need **finite representation** of quasimodels for decidability

Weak Quasimodels to the Rescue

Main Ideas

- Define **weak quasimodels** with weakened saturation conditions

Weak Quasimodels to the Rescue

Main Ideas

- Define **weak quasimodels** with weakened saturation conditions
- Show that quasimodels can be reconstructed from **bounded-size** weak quasimodels

Overview of Results for $\mathbf{K}_n$ and $\mathbf{S5}_n$ Validity

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$	$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$
S5	$=$	coNExp	coNExp	2Exp	coNExp	coNExp	2Exp
$\mathbf{S5}_n$, $n \geq 2$	$=$	coNExp	coNExp	2Exp	undecidable		
$\mathbf{K}_n$	$=$	coNExp	coNExp	2Exp	undecidable		
	$\subseteq$	PSpace	coNExp	2Exp	?		
$\mathbf{K}_{*n}$, $\mathbf{LTL}^{(\Diamond)}$	$=$	Σ^1_1					
	$\subseteq$	undecidable					
$\mathbf{K}f_{*n}$, $\mathbf{LTL}f^{(\Diamond)}$	$=$	undecidable					
	$\subseteq$	decidable, Ackermann-hard					

One-Variable Fragment $\mathbf{K}_n/\mathbf{S5}_n$-Validity in Constant Domains

For $\mathbf{K}_n$ and $\mathbf{S5}_n$**-validity** in $\mathrm{Q}^{1=}\mathrm{ML}_c$ with **constant** domains

- guess-and-check **exponential-size** weak quasimodels
- **coNExpTime upper bound**; matching lower bound from constant-domain products / one-variable **K** and **S5** without equality and constants

Theorem

With constant domains $\mathbf{K}_n$ and $\mathbf{S5}_n$-validity in $\mathrm{Q}^{1=}\mathrm{ML}_c$ are coNExpTime*-c.*

- *in fact, every satisfiable sentence is satisfiable in a frame of exponential size*

Guarded Fragment $\mathbf{K}_n/\mathbf{S5}_n$-Validity in Constant/Expanding Domains

For $\mathbf{K}_n$ and $\mathbf{S5}_n$**-validity** in $\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_c$ with **constant/expanding** domains:

- **enumerate quasimodels** and **check realisable** quasistates in **double exp. time**
- 2**ExpTime upper bound**; matching lower bound from plain GF

Theorem

With constant/expanding domains, $\mathbf{K}_n$ *and* $\mathbf{S5}_n$*-validity in* $\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_c$ *are* 2EXPTIME*-c.*

Two-Variable Fragment $\mathbf{K}_n/\mathbf{S5}_n$-Validity in Constant/Expanding Domains

For $\mathbf{K}_n$ and $\mathbf{S5}_n$**-validity** in $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota$ with **constant/expanding** domains:

- introduce **weak pre-quasimodels** replacing multiset of weak runs with a bounded-from-above set of **locally saturated weak runs**
- **encode quasistates** and other constraints in decidable **Presburger arithmetic** extended with infinity ($\aleph_0$) and exploit NExpTime upper bound for C^2
- **coNExpTime upper bound**; matching lower bound from plain C^2

Theorem

With constant/expanding domains, $\mathbf{K}_n$ and $\mathbf{S5}_n$-validity in $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$ are coNExpTime-c.

Decidability with Expanding Domains

Expanding vs Constant Domains

- Recall: reasoning in **expanding** domains can be **reduced** to **constant** domains
- However: expanding domains sometimes **simpler** than constant domain case
- Quasimodel and weak quasimodel constructions work for expanding domains

Decidability with Expanding Domains

Expanding vs Constant Domains

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- Quasimodel and weak quasimodel constructions work for expanding domains

Affected Fragments

Under expanding domains, life is (a bit) easier for

- validity/global conseq. in fragments with **transitive closure** & **no infinite chains**
- $\mathbf{K}_n$-validity in **one-variable fragment**

From Validity to Global Consequence with Transitive Closure

Definition ($\mathbf{K}_{*n}$ and $\mathbf{Kf}_{*n}$ Frames)

- Modalities $A = A_0 \cup \{*\}$
- $\mathbf{K}_{*n}$: frames with **transitive closure** R_* of $\bigcup_{a \in A_0} R_a$ (interpreting $\Diamond_*$)
- $\mathbf{Kf}_{*n}$: frames where R_* has **no infinite ascending chain** $w_i R_* w_{i+1}$, for all $i \geq 0$

Lemma

For all fragments $\mathcal{L}$ and $\mathcal{C} \in \{\mathbf{K}_{*n}, \mathbf{Kf}_{*n}\}$, with both constant and expanding domains, global $\mathcal{C}$-consequence in $\mathcal{L}$ is polytime-reducible to $\mathcal{C}$-validity in $\mathcal{L}$

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Proof (Idea).

φ global $\mathcal{C}$-consequence of Γ iff $(\bigwedge \Gamma \wedge \Box_* \bigwedge \Gamma) \rightarrow \varphi$ $\mathcal{C}$-valid for $\mathcal{C} \in \{\mathbf{K}_{*n}, \mathbf{Kf}_{*n}\}$ □

Overview of Results for Expanding Domains

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$\mathrm{Q}^{1=}\mathrm{ML}_\iota$	$\mathrm{C}^2_{\boxed{1}}\mathrm{ML}_\iota$	$\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_\iota$	$\mathrm{Q}^{1=}\mathrm{ML}_\iota$	$\mathrm{C}^2_{\boxed{1}}\mathrm{ML}_\iota$	$\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_\iota$
S5	$=$	coNExp	coNExp	2Exp	coNExp	coNExp	2Exp
S5$_n$, $n \geq 2$	$=$	coNExp	coNExp	2Exp	undecidable		
K$_n$	$=$	coNExp	coNExp	2Exp	undecidable		
	$\subseteq$	PSpace	coNExp	2Exp	?		
K$_{*n}$, **LTL**$^{(\diamond)}$	$=$	Σ^1_1					
	$\subseteq$	undecidable					
Kf$_{*n}$, **LTL*f***$^{(\diamond)}$	$=$	undecidable					
	$\subseteq$	decidable, Ackermann-hard					

Decidability with Transitive Closure in Expanding Domains

Theorem

With expanding domains, $\mathbf{K}\boldsymbol{f}_{*n}$-validity in $\mathrm{C}^2_{\boxed{1}}\mathrm{ML}_c$ and $\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_c$ are decidable

Decidability with Transitive Closure in Expanding Domains

Theorem

With expanding domains, $\mathbf{Kf}_{*n}$-validity in $\mathrm{C}^2_{\boxed{1}}\mathrm{ML}_c$ and $\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_c$ are decidable

Proof (Idea).

Relies on weak quasimodels and shows, using Dickson's Lemma, a non-primitive recursive bound on their size □

$\mathbf{K}_n$-Validity in One-Variable Fragment

Theorem

For expanding-domain models, $\mathbf{K}_n$-validity in $\mathsf{Q}^{1=}\mathsf{ML}_c$ is PSPACE*-complete*

$\mathbf{K}_n$-Validity in One-Variable Fragment

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For expanding-domain models, $\mathbf{K}_n$-validity in $\mathrm{Q}^{1=}\mathrm{ML}_c$ is PSPACE*-complete*

Proof (Idea).

- Upper bound: define a non-deterministic recursive function that checks the existence of a quasimodel for a formula in polynomial space
- Lower bound: from the underlying (propositional) modal logic $\mathbf{K}_n$ □

Overview

1 Introduction

2 Preliminaries

3 Results
   Related Formalisms and Reductions
   Quasimodels and Weak Quasimodels
   Temporal Logics

4 Conclusion

Frames for Temporal Logics

Definition (Temporal Frame Classes)

- **LTL**$^{\Diamond}$: $\{(\mathbb{N}, <)\}$, with standard **strict linear order** $<$ (interpreting $\Diamond$)
- **LTL*f***$^{\Diamond}$: $\{\,(\{0, \ldots, n\}, <) \mid n \in \mathbb{N}\,\}$, with $<$ restricted to $\{0, \ldots, n\}$
- **LTL**: $\{(\mathbb{N}, <, S)\}$, with **successor** relation $S = \{(i, i+1) \mid i \in \mathbb{N}$ (interpreting $\bigcirc$)
- **LTL*f***$^{\Diamond}$: $\{\,(\{0, \ldots, n\}, <, S) \mid n \in \mathbb{N}\,\}$, with $<$ and S restricted to $\{0, \ldots, n\}$

Overview of Results for Temporal (and Transitive Closure) Logics

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$	$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$
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$\mathbf{K}_{*n}$, $\mathbf{LTL}^{(\Diamond)}$	$=$	Σ^1_1					
	$\subseteq$	undecidable					
$\mathbf{Kf}_{*n}$, $\mathbf{LTLf}^{(\Diamond)}$	$=$	undecidable					
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(Un-)Decidability of Temporal Fragments

Theorem

In $\mathsf{Q}^{1=}\mathsf{LTL}_\iota$ *and* $\mathsf{Q}^{1=}\mathsf{LTL}_\iota^\Diamond$ *with*

- **constant domains**: $\begin{cases} \textbf{LTL}\text{-}\textit{validity} & \Sigma_1^1\text{-}\textit{complete} \\ \textbf{LTL}\boldsymbol{f}\text{-}\textit{validity} & \textit{undecidable and co-r.e.} \end{cases}$
- **expanding domains**: $\begin{cases} \textbf{LTL}\text{-}\textit{validity} & \textit{undecidable and r.e.} \\ \textbf{LTL}\boldsymbol{f}\text{-}\textit{validity} & \textit{decidable but Ackermann-hard} \end{cases}$

(Un-)Decidability of Temporal Fragments

Theorem

In $Q^{1=}LTL_\iota$ *and* $Q^{1=}LTL_\iota^\diamond$ *with*

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Proof (Idea).

Adapt known results from products/1-variable temporal logics with difference operator

- Lower bounds: undecidable/Ackermann-hard $Q^{1\neq}LTL \rightsquigarrow Q^{1=}LTL_\iota$
- Upper bounds: $Q^{1=}LTL_\iota \rightsquigarrow Q^{1\neq}LTL$ decidable/undecidable r.e./co-r.e./in Σ_1^1 □

From Temporal to Modal Logics with Transitive Closure

Theorem (Polytime Reduction from Temporal to Modal with Transitive Closure)

In $\mathrm{Q}^{1=}\mathrm{LTL}_\iota$, $\mathrm{C}^2_{\boxed{1}}\mathrm{LTL}_\iota$, $\mathrm{GF}^{=}_{\boxed{1}}\mathrm{LTL}_\iota$, *with both constant and expanding domains*

- **LTL**-*validity is polytime-reducible to* $\mathbf{K}_{*n}$-*validity*
- **LTL*f***-*validity is polytime-reducible to* $\mathbf{K}\boldsymbol{f}_{*n}$-*validity*

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Proof (Idea).

Adapt reduction from product $\mathbf{LTL} \times L$ to $\mathbf{K}_{*n} \times L$ □

From Temporal to Modal Logics with Transitive Closure

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Proof (Idea).

Adapt reduction from product $\mathbf{LTL} \times L$ to $\mathbf{K}_{*n} \times L$ □

Remark

With (un-)decidability results above, implies lower bounds for $\mathbf{K}_{*n}$ / $\mathbf{K}\boldsymbol{f}_{*n}$

Decidability with Expanding Domains over Finite Traces

Theorem (Decidability on Finite Traces with Expanding Domains)

For expanding-domain models, **LTL*f*** *-validity in* $\mathsf{C}^2_{\boxed{1}}\mathsf{LTL}_\iota$ *and* $\mathsf{GF}^=_{\boxed{1}}\mathsf{LTL}_\iota$ *is decidable.*

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Proof (Idea).

Reduce to $\boldsymbol{Kf}_{*n}$-validity with expanding domains and apply decidability result □

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Summary of Results

Recap

Established **decidability** and **tight complexity bounds** for monodic fragments with:

- non-rigid designators (non-rigid constants and definite descriptions)
- non-trivial counting (equality or counting quantifiers)
- both constant and expanding domains
- several classes of frames ($\mathbf{K}_n$, $\mathbf{S5}_n$, $\mathbf{K}_{*n}$, $\mathbf{K}\boldsymbol{f}_{*n}$, linear time)

Summary of Results

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$	$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$
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Discussion and Future Work

Description Logic Applications

- Powerful positive results for modal/temporal **DLs based on** $\mathcal{ALCQHIO}^u$
- **(Temporal) ontology-mediated query answering** with NRDC features
- **Other expressive DLs** not yet considered in modal/temporal contexts

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Other First-Order Extensions

- **Guarded negation fragment**
- **Fluted fragments**
- **Two-variable fragment** with **semantically-constrained relations**, e.g., transitive or equivalence relations

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Other Modal Logic Approaches

- **Bundled fragments**: restricted modality/quantifier patterns ($\exists x \Diamond$, $\Diamond \forall x$)
- **Term modal logics**: modal operators indexed by non-rigid agent names

Some of Our Papers

Check Other References Therein!

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Thank You!

Questions?