On Dynamics of Continuous and Baire one Functions

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Definitions:

- For any integer $n \geq 1$, f^n denotes the n^{th} iteration of f.

- For each $x \in X$, we call $O(x,f) = \{f^n(x)\}_{n=0}^{\infty}$ the orbit or trajectory of x under f.

- $$F(f) = \{x : f(x) = x\}$$

is the set of fixed points of f.

- $$P(f) = \{x : f^n(x) = x \text{ for n} \in \mathbb{N}\}$$

is the set of periodic points of f,

- $$R(f) = \{x : \lim_{k\to\infty} f^{n_k}(x) = x\}$$

is the set of recurrent points of f.

• $\omega(x,f)$ the ω-limit set of f generated by x is the set of sub-sequential limits of $O(x,f)$. The set $\Lambda(f)=\cup_{x\in X}\omega(x,f)$ is called the set of ω-limit points of f.

• The point $x\in X$ is called a chain recurrent point of f, if for each $\epsilon>0$ there exits a sequence of points $\{x_k\}_{k=0}^{n}$ such that $x_0=x=x_n$ and

$$\rho(f(x_i),x_{i+1})<\epsilon,\qquad i=0,1,\cdots,n-1.$$

$$F(f)\subseteq P(F)\subseteq R(f)\subseteq \Lambda(f)\subseteq CR(f)$$

• Dynamics of continuous functions and the structure of the above sets related to $f \in C(X, X)$ has been studied extensively. In the literature there are several definitions of chaos for continuous functions. Common to all, is the idea that points arbitrarily close together can have orbits or ω-limit sets that spread out. For example, the notion of topological entropy and Li-York chaos make explicit use of the separation of trajectories, while Devaney chaos incorporates sensitivity and requires that nearby points must generate distinct ω-limit sets.

Bruckner-Ceder chaos on the unit interval
Pacific J. Math., (1992)

Let $f \in C(I, I)$. Consider the family of ω-limit sets of f, $\{\omega(x, f) : x \in I\}$, endowed with Hausdorff metric. They studied the continuity of the map $\omega_f : x \to \omega(x, f)$ and showed that ω_f is rarely continuous. In fact they proved that

$$h(f) > 0 \quad \Longrightarrow \quad \omega_f \notin B_1 \Longrightarrow f \text{ is Li} - \text{York chaotic},$$

with none of these implications being reversible.

• **D'Aniello, E., Steele, T. H.;** Chaotic behavior of the map $\omega_f : x \to \omega(x, f)$, Cent. Eur. J. Math. 12 (2014).

In this paper the authors address various concepts of chaos, including topological entropy, Li-Yorke chaos, Devaney chaos, and Bruckner-Ceder chaos on the unit interval. They also study the continuity of $\omega_f : x \to \omega(x, f)$ for continuous Cantor functions.

A functions $f \in \mathcal{E}(X, X)$ typically has a certain property if the set of those functions which do not have this property is of first category in $\mathcal{E}(X, X)$.

• Typically continuous self-maps of an interval have σ-perfect, measure zero sets of periodic points.

• **(A.A., Int. J. Math. Math. Sci. 2013)**
Typically members of $C^1(I, I)$ have a countable set of periodic points.

• **(N. Franzova, Acta Math. Univ. Comenian., 1991)**

Typical continuous function have sets of chain recurrent points of zero Lebesgue measure.

• **(Agronsky, Bruckner & Laczkovick , London J. of Math., (1989))**

Typical continuous functions on $[0, 1]$ have no stable (attracting) points and most points in $[0, 1]$ are attracted to Cantor sets. Moreover, if A is any prescribed residual subset of $[0, 1]$ (which could be a null set) then for a typical $f, \Omega(f)$, the set of non-wandering points of f, is a nowhere dense subset of A.

Dynamics of non-continuous functions:

Researchers have studied the dynamics of some non-continuous functions. For example:

K. Kellum, RAE, (1989) Iterates of almost continuous functions and Sharkowskii's theorem
P. Szuca, (2003) Sharkowski's theorem holds for some discontinuous functions
R. J. Pawlak, Colloq. Math.(2009) On the entropy of Darboux functions.
T. Natkaniec, P. Szuca, Colloq. Math. (2010) On Pawlak's problem concerning entropy of almost continuous functions

Sub-classes of Baire one functions

• $\mathcal{A}$ is the class of approximately continuous functions

• Δ is the class of derivatives

• DB_1 is the class of Darboux Baire one functions

• B_1 is the class of Baire one functions

• Note that the corresponding bounded classes $\mathcal{E}(I, I)$ endowed with

$$\rho(f,g) = \sup_{x\in I} |f(x) - g(x)| \text{ for } f,\ g \in \mathcal{E}(\mathrm{I},\mathrm{I})$$

Form Banach Spaces; and

$$b\mathcal{A} \subset b\Delta \subset bDB_1 \subset bB_1$$

Question 1. What can be said about dynamics of $f \in \mathcal{E}(I, I)$, where $\mathcal{E}(I, I)$ is one of the above sub-classes of Baire one functions?

Question 2. Do typically elements of these sub-classes of bB_1 functions have small sets of fixed points, periodic points, etc. in some sense; for example in the sense of cardinality, category, measure or Hausdorff dimension?

If not what can be said about the size of these sets?

Observation:

• Let $\mathcal{E}(X)$ be a class of self-maps of X and $\mathcal{C}(X) \subset \mathcal{E}(X)$, where $\mathcal{C}(X)$ is the set of continous self maps of X.

In studying, dynamical systems of non-continuous members of $\mathcal{E}(X)$ we encounter difficulties due to the luck of the following properties;

• $\mathcal{E}(X)$ being closed under composition of functions;

• $\mathcal{E}(X)$ being closed under uniform limits; and

• If $\{f_k\} \subset \mathcal{E}(X)$ converges to f

(i) pointwise, then $f_k^n(x)$ converges to $f^n(x)$;
(ii) uniformly, then f_k^n converges to f^n uniformly.

- The iterates of a Baire one function are not necessarily Baire 1.

Example: Let $Q = \{q_n\}_{n=1}^{\infty} \subset (0,1]$ be an enumerations of rational numbers with $q_1 = 1$, define:

$$f(x) = \begin{cases} \frac{\sqrt{2}}{n}, & x = q_n, x \neq 0 \\ 1, & x = 0, \\ 0, & x \in \mathbb{R} \setminus Q. \end{cases}$$

Then we have

$$f^2(x) = f \circ f = \begin{cases} 0, & x = q_n, x \neq 0 \\ \sqrt{2}, & x = 0, \\ 1, & x \in \mathbb{R} \setminus Q. \end{cases}$$

$$f^3(x) = f\circ f\circ f = \begin{cases} 1, & x = q_n, x \neq 0 \\ 0, & x = 0, \\ \sqrt{2}, & x \in \mathbb{R}\setminus Q. \end{cases}$$

$$f^4(x) = f\circ f\circ f\circ f = \begin{cases} \sqrt{2}, & x = q_n, x \neq 0 \\ 1, & x = 0, \\ 0, & x \in \mathbb{R}\setminus Q. \end{cases}$$

That is

$$\begin{aligned} f^2 = f^5 = f^8 &= \cdots, \\ f^3 = f^6 = f^9 &= \cdots, \\ f^4 = f^7 = f^{10} &= \cdots. \end{aligned}$$

Thus $f^n \notin B_1$ for all $n \geq 2$.

• **(Fenecios, J.; Cabral, E., J. Indones. Math. Soc. 18 (2012))**

Def. A function f is called left compositor (right compositor) if $g \circ f \in B_1$ (if $f \circ g \in B_1$) for every $g \in B_1$..

$k-$continuous functions are exactly the right Baire one compositor.

Def. $f : \mathbb{R} \to \mathbb{R}$ is $k-$continuous if for every $\epsilon : \mathbb{R} \to (0, \infty)$, there exist a $\delta : \mathbb{R} \to (0, \infty)$ such that $\mid f(x) - f(y) \mid < \min\{\epsilon(x),\ \epsilon(y)\}$ when $\mid x - y \mid < \min\{\delta(x), \delta(y)\}$.

A.A. Theorem. Let $f \in B_1(I, I)$ have finite range. Then f is a k-continuous function.

T. H. Steele, Theorem. There exists a residual subset S of bB_1 such that if $f \in S$, then the function $f^n \in bB_1$ for all natural numbers n.

A.A., Theorem. Let D be a residual subset of $[0, 1]$. Then there exists a dense family E of k-continuous functions and a dense G_δ family $\mathcal{F} \subset bB_1$ with the following properties;

a) for each $f \in E$, the range of f is a finite subset of D.

b) For each $n \geq 1$ and $f \in \mathcal{F}$, there exits a sequence $\{f_{n_k} = h_{n_k}(g_{n_k})\}_{k=1}^{\infty} \subset E$ such that $f_{n_k} \to f^n$ uniformly, and the range of f_{n_k} is a finite subset of D.

Hanson, B.; Pierce, P.; Steele T. H. Dynamics of typical B_1 functions on a compact n-manifold. Aequationes Math. (2019).

• For a typical self-map $f \in bB_1$ on an n-manifold M, the following hold:

(i) $f : \Lambda(f) \to \Lambda(f)$ is a bijection, and $\Lambda(f)$ is closed;

(ii) the Haussdorff dimension of $\Lambda(f)$ is zero;

(iii) the collection of ω-limit sets $\Omega(f)$ is closed in the Haussdorf metric space.

P. Y. Lee, W. K. Tang, and D. Zhao (Proc. of AMS, 1992)

$\epsilon - \delta$ **Definition of Baire one Functions:**

Let (X, ρ_1) and (Y, ρ_2) be two metric spaces and $\mathbb{R}^+$ be the set of positive reals. A function $f : X \to Y$ is a B_1 function if for any positive number ϵ, there is a positive function $\delta : X \to \mathbb{R}^+$ such that for every $x, x_0 \in X$,

$$\rho_1(x, x_0) < \min\{\delta(x), \delta(x_0)\} \Longrightarrow \rho_2(f(x), f(x_0)) < \epsilon.$$

Research Interest: The equi-continuity of iterates of continuous functions is pivotal in understanding their dynamics.. During my sabbatical in 2018, my exploration of the dynamics within the subclasses of Baire one functions prompted a deeper investigation into the concept of equi-continuity. Surprisingly, my search for a useful generalization proved fruitless. However, I then realized that we could utilize the $\epsilon - \delta$ definition provided for B_1 functions to establish the notion of equi-B_1 property. This realization felt so straightforward that I anticipated it should have been observed earlier. My search for the generalization to support my anticipation was not successful, however my anticipation turned out to be correct. In fact, the equi-B_1 property of families of functions is defined in the article:

Dominique Lecomte, "How can we recover Baire class one functions?", Mathematika 50 (2003).

Lecomte D. (2003), (A. A. 2018) Definition. The family $\mathcal{F} \subset B_1(X.Y)$ is said to be equi-Baire one at x_0, if for $\epsilon > 0$ there exists a function $\delta : X \to \mathbb{R}^+$ such that for all $f \in \mathcal{F}$,

$$\rho_2(f(x), f(x_0)) \leq \epsilon, \text{ if } \rho_1(x, x_0) < \min\{\delta(x), \delta(x_0)\}.$$

• $\mathcal{F}$ is said to be equi-Baire one if for $\epsilon > 0$, there exists a function $\delta : X \to \mathbb{R}^+$ such that for all x and y in X, and all $f \in \mathcal{F}$,

$$\rho_2(f(x), f(y)) \leq \epsilon, \text{ if } \rho_1(x, y) < \min\{\delta(x), \delta(y)\}.$$

Theorem Let (X, ρ_1), (Y, ρ_2) be complete separable metric spaces. Then the following statements are equivalent:
(i) The family $\mathcal{F} \subseteq B_1(X, Y)$ is equi-Baire one;
(ii) for every $\epsilon > 0$ there exists a sequence of closed sets $\{X_i\}_{i=1}^{\infty}$ such that
$X = \cup_{i=1}^{\infty} X_i$ and $\mathcal{O}_f(X_i) \leq \epsilon$ for all $f \in \mathcal{F}$ and $i \in \mathbb{N}$, where

$$\mathcal{O}_f(X_i) = \sup\{\rho_2(f(x), f(y)) : x, y \in X_i\}.$$

- **(A.A., (2020))**

Example 1: A family of Baire one functions that is not equi-Baire one.

Let $\mathbb{Q} = \{r_i\}_{i=1}^{\infty}$ be an enumeration of rationals. Then the family

$$\mathcal{F} = \{f_n(x) : n = 1, 2, \cdots\}$$

where,

$$f_n(x) = \begin{cases} 2, & x = r_i, \ i = 1, 2, \cdots, n \\ 1 & \text{elsewhere.} \end{cases}$$

is a family of Baire one functions that is not equi-B_1.

Example 2: A family $\mathcal{F} \subset C(I, I)$ that is equi-B_1, but it is not equi-continuous.

The family $\mathcal{F} = \{g_k(x)\}_{k=1}^{\infty}$ where

$$g_k(x) = \begin{cases} k^2 x, & 0 \leq x \leq \frac{1}{k}, \\ \frac{1}{x}, & \frac{1}{k} \leq x \leq 1. \end{cases}$$

is Equi-B_1 but not equi-continuous.

Example. There exists a sequence of Baire one functions $\{f_n\}$ defined on a compact metric space X which is uniformly convergent to a function f and for each $n \geq 1$, $\{f_n^m\}_{m\geq 1}$ is equi-B_1, but the family $\{f^m\}_{m\geq 1}$ is not equi-B_1.

Let $\mathbb{Q}$ be the set of rational numbers in $[0,1]$, $A = \{\frac{1}{i}\}_{i=2}^{\infty}$, $B = \{0,1\}$ and $\{r_i\}_{i=1}^{\infty}$ be an enumeration of $\mathbb{Q} \setminus (A \cup B)$. Define

$$f_m(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = r_n,\ n \leq m, \\ 1 - x & \text{if } x = 0 \text{ or } x = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = r_n, n = 1, 2, \cdots , \\ 1 - x & \text{if } x = 0 \text{ or } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for each $m \geq 1$, f_m has finitely many points of discontinuity, hence $\{f_m\}_{m=1}^{\infty} \subset bB_1$. Also

$$\| f_m - f \| \leq \sup\{| f_m(r_n) - f(r_n) | = \frac{1}{n+1}, \text{ where } n > m\} \leq \frac{1}{m+1},$$

thus f_m converges uniformly to f, hence $f \in bB_1$. For each $k \geq 1$, we have

$$f_m^{2k}(x) = \begin{cases} 0 & \text{if } x = r_n,\ n \leq m, \\ x & \text{if } x = 0 \text{ or } x = 1, \\ 1 & \text{otherwise,} \end{cases} ,$$

$$f_m^{2k+1}(x) = \begin{cases} 1 & \text{if } x = r_n,\ \ n \leq m, \\ 1 - x & \text{if } x = 0 \text{ or } x = 1, \\ 0 & \text{otherwise,} \end{cases},$$

$$f^{2k}(x) = \begin{cases} 0 & \text{if } x = r_n, n = 1, 2, \cdots, \\ x & \text{if } x = 0 \text{ or } x = 1, \\ 1 & \text{otherwise,} \end{cases}, \text{ and}$$

$$f^{2k+1}(x) = \begin{cases} 1 & \text{if } x = r_n, n = 1, 2, \cdots, \\ 1 - x & \text{if } x = 0 \text{ or } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for each $k \geq 1$, f_m^k has finitely many points of discontinuity, hence $\{f_m^k\}_{m\geq 1} \subseteq bB_1$. Thus $\mathcal{F}_1 = \left\{f_m^k\right\}_{k\geq 1} = \{f_m(x), f_m^2(x), f_m^3(x)\}$ is a finite family of Baire one functions, hence it is an equi-B_1 family of functions. However, the family $\mathcal{F}_2 = \left\{f^k(x)\right\}_{k\geq 1} = \{f(x), f^2(x), f^3(x)\}$,
$f^{2k}(x) = 1 - f^{2k+1}(x)$, and
$f^{2k+1}(x) = g(x) - h(x)$, where

$$h(x) = \begin{cases} 1 & \text{if } x = \frac{1}{i}, i = 2, 3, \cdots, \\ x & \text{if } x = 0 \text{ or } x = 1, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and } g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise}. \end{cases}$$

The set of discontinuity points of h is $D = A \cup B$ that is countable, so $h \in B_1$, $g \in B_2 \setminus B_1$, hence $\{f^{2l+1} = g - h, f^{2l} = 1 - f^{2l+1}\} \subset B_2 \setminus B_1$. Thus $\mathcal{F}_2 = \{f^m\}_{m \geq 1}$ is not an equi$-B_1$ family of functions.

Definition: Ler $\mathcal{F} = \{f^n\}_{n\geq 1} \subset B_1$. Then

• f is said to be orbitally equi-Baire one if and only if $\mathcal{F}$ is an equi-Baire one family.

Definition. Let $\mathcal{A}$ be a class of self-maps of a metric space (X, ρ) and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathcal{A}$. Then $\{f_n\}_{n=1}^{\infty}$ is said to be uniformly orbitally convergent to a map $f : X \to X$, if for for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(f_n^m(x), f^m(x)) < \epsilon$, for all $x \in X$, for all $m \in \mathbb{N}$ and for all $n \geq k$.

A.A., Theorem. Let (X, ρ) be a compact metric space and suppose that $\{f_n\}_{n=1}^{\infty} \subset B_1$ be a sequence such that for each n, the family $\{f_n^m\}_{m=1}^{\infty}$ is equi-B_1. If $\{f_n\}_{n=1}^{\infty}$ converges uniformly orbitally to f, then f is orbitually equi-B_1.

$$\mathcal{F}_u(I) = \{\{f_n\}_{n=1}^{\infty} \subset B_1(I) : f_n \to f, \text{ uniformly on } I\}$$

$$\mathcal{F}_{eq}(I) = \{\{f_n\}_{n=1}^{\infty} \subset B_1(I) : \{f_n\}_{n\geq 1} \text{ is equi}-B_1, \text{ and } f_n(x) \to f(x)\}$$

$$\mathcal{F}_{p.w.}(I) = \{\{f_n\}_{n=1}^{\infty} \subset B_1(I) : f_n(x) \to f(x), \ x \in I, \text{and } f \in B_1, x \in I\}$$

$$\mathcal{F}_u(I) \subsetneqq \mathcal{F}_{eq}(I) \subsetneqq \mathcal{F}_{p.w.}(I)$$

Bruckner and Ceder, Paificc. J. Math.,

Theorem. For $f \in C(I, I)$, the following are equivalent:

(1) ω_f is continuous.

(2) $\{f^n\}_{n=1}^{\infty}$ is eqicontinuous.

(3) ω_{f^2} is continuous.

(4) $\mathrm{Fix}(f^2) = \cap_{n=1}^{\infty} f^n(I)$.

(5) $\mathrm{Fix}(f^2)$ is connected and for all x, $\{f^{2n}(x)\}_{n=1}^{\infty}$ converges to a point of $\mathrm{Fix}(f^2)$.

(6) $\mathrm{Fix}(f^2)$ is connected.

(7) ω_f is lower semi-continuous.

(8) ω_f is upper semi-continuous.

Equi-Baire one property is a generalization of equicontinuity, then it is natural to ask the following question:

Question: Does Equi-B_1 property of iterates of $f \in B_1(I, I)$ give us similar results to its continuous counterpart given in the previous theorem?

Theorem. Let X be a compact metric space and let $f \in C(X, X)$. Then ω_f is of Borel class 2.

T.H. Steele, Theorem. For a typical $f \in bB_1$, the map $\omega_f : I \to \mathcal{K}$ given by $x \to \omega(x, f)$ is Baire-3.

Question. What kind of results similar to various parts of the above theorem can be obtained for the class of Baire one self-maps of intervals with equi-B_1 iterates?

Theorem. For typical $f \in B_1(I, I)$, the family $\{f^n\}_{n=1}^{\infty}$ is equi-B_1.

Theorem. Let $f : I \to I$ be a function such that $\{f^n\}_{n=1}^{\infty}$ is equi-Baire one. Then the function $\omega_f : x \to \overline{\omega(x, f)}$ is a Baire one function on I.

Corollary. For a typical $f \in B_1(I, I)$, $\omega_f : x \to \overline{\omega(x, f)}$ is a Baire one function.

The condition of the equi-B_1 property of iterates of f in the above Theorem is a necessary condition.

Example. There exists a function $f : I \to I$ such that $\{f^n\}_{n\geq 1} \subset B_1(I, I)$, the family $\{f^n\}_{n\geq 1}$ is not equi-B_1, and $\omega_f : x \to \overline{\omega(x, f)}$ is not a Baire one function.

Let $A = \{\frac{\sqrt{2}}{m}\}_{m\geq 1}$, $B = (0, 1] \setminus (A \cup \mathbb{Q})$, and $\mathbb{Q} = \{r_n\}_{n\geq 1}$ be an enumeration of rational numbers in the open interval $(0, 1)$. We define

$$f(x) = \begin{cases} \frac{\sqrt{2}}{n+1} & \text{if } x = r_n,\ n \geq 1, \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x \in B, \\ \frac{\sqrt{2}}{m+1} & \text{if } x = \frac{\sqrt{2}}{m}, m = 2, 3, \cdots . \end{cases}$$

Then $f \in bB_1$ and for $k \geq 1$,

$$f^{2k}(x) = \begin{cases} \frac{\sqrt{2}}{n+2k} & \text{if } x = r_n, n \geq 1, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \in B, \\ \frac{\sqrt{2}}{m+2k} & \text{if } x = \frac{\sqrt{2}}{m}, m = 2, 3, \cdots , \end{cases}$$

and

$$f^{2k+1}(x) = \begin{cases} \frac{\sqrt{2}}{n+2k+1} & \text{if } x = r_n, \ \ n \geq 1, \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x \in B, \\ \frac{\sqrt{2}}{m+2k+1} & \text{if } x = \frac{\sqrt{2}}{m}, m = 2, 3, \cdots . \end{cases}$$

It is clear that

$$\omega(x,f)=\begin{cases}\{0\}, & \text{when } x=r_n, \text{ for some } n,\\ \{0,1\}, & \text{when } x\in B,\end{cases}$$

and

$$d_H(\overline{\omega(x,f)},\overline{\omega(r_n,f)})>\frac{1}{2}, \quad \text{for } x\in B.$$

Hence $\omega_f : x \to \overline{\omega(x,f)}$ is not Baire one on I.

M. Balcerzak, O. Karlova & P. Szuca; Equi-B_1 families of functions, Topol. Appl. 305 (2022)

Theorem. Let X be a separable metric space and (f_n) be a pointwise bounded sequence of functions $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, that form an equi-B_1 family. Then there exists a subsequence (f_{k_n}) which is pointwise convergent on X to a Baire 1 function.

Note: Let $f : \mathbb{R} \to \mathbb{R}$ be a Baire one function with an equi-B_1 iterates, then $\omega_f : \mathbb{R} \to \mathbb{R}$ defined by $\omega_f(x) = \overline{\omega(x, f)}$ is a Baire one function and also there exists a subsequence f^{k_n} that converges to a Baire one function. One might ask the following question:

Question: Assume $(f^n) \subset B_1$ and $\omega_f \in B_1$. What additional conditions imposing on the family $\{f^n\}_{n=1}^{\infty}$ will guarantee that $\{f^n\}_{n=1}^{\infty}$ is equi-B_1?

Summary of some known results for B_1 functions

Theorem. There exists $\mathcal{T}$ a residual subset of $bB_1(I)$ such that for any $f \in \mathcal{T}$, the following holds:

• The range of f is contained in the sets of points of continuity of f, and f is a one-to-one function.

• For any $x \in I$, the ω-limit set $\omega(x, f)$ is contained in the set of continuity points of f.

• For any $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $f^m(I) \subseteq B_\epsilon(\Lambda(f))$ whenever $m \geq M$. Moreover, $f : \Lambda(f) \to \Lambda(f)$ is a bijection and $\Lambda(f)$ is closed.

• The Hausdorff dimension $dim_H(\overline{\Lambda(f)}) = 0$.

• The collection of $\omega-$limit sets $L(f)$ is closed in the Hausdorff metric space.

• If x is a point at which f is continuous, the (x, f) is a point at which the map $\omega : I \times \mathcal{T} \to \mathcal{K}$ given by $(x, f) \to \omega(x, f)$ is continuous.

• The n-fold iterates f^n is an element of bB_1 for all $n \in \mathbb{N}$.

• The function f is not chaotic in the sense of Devaney, nor it is chaotic in the sense of Li-York.

Theorem. Let μ be an arbitrary continuous Borel measure on $I = [0, 1]$. Then for a typical $f \in bB_1$,

- $\mu(f^{-1}(F(f))) = 0$.
- $\overline{CR(f)} = CR(f)$.
- $\mu(\overline{CR(f)}) = 0$.
- the family $\{f^n\}_{n=1}^{\infty}$ is equi-B_1.
- $\omega_f : x \to \overline{\omega(x, f)}$ is a Baire one function.
- the set function $\omega_f : x \to \omega(x, f)$ has a Baire one selection.

• **Conclusion:** Generally, one thinks of continuous functions are better dynamically behaved than Baire 1 functions. It seems, this is not the case. In fact, a typical Baire 1 function tend to have much simpler dynamics than its continuous counterpart.

END

Thank You For Listening

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