Characterizing Lipschitz continuity through modes of convergence for measurable functions

Nuno J. Alves

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Contents

Motivation:

From convergence in relative entropy to convergence almost everywhere

Let $(X, \mathbb{X}, \mu)$ be a measure space.

Let $p > 1$ and consider the function $H : [0, \infty) \to [0, \infty)$ given by

$$H(f) = f^p$$

Define the relative entropy of H as

$$H(g|f) = H(g) - H(f) - H'(f)(g - f)$$

Let $f_n, f : X \to \mathbb{R}$ be measurable.

Question

$$\int_X H(f_n|f)\,d\mu \to 0 \quad \overset{?}{\Longrightarrow} \quad \exists \textit{ subsequence } (f_{k_n}) \textit{ that converges to } f \textit{ a.e.}$$

Question : $\int_X H(f_n|f)\,d\mu \to 0 \quad \overset{?}{\Longrightarrow} \quad \exists$ subsequence (f_{k_n}) that converges to f a.e.

This is trivial in the case $p = 2$, since for $H(f) = f^2$ the relative entropy is

$$\begin{aligned} H(g|f) &= H(g) - H(f) - H'(f)(g-f) \\ &= (g-f)^2 \end{aligned}$$

and so

$$\int_X H(f_n|f)\,d\mu = \|f_n - f\|_2^2$$

$$H(f) = f^p, \quad p > 1$$
$$H(g|f) = g^p - f^p - pf^{p-1}(g - f)$$

Lemma (C. Lattanzio, A. E. Tzavaras, *SIAM J. Math. Anal.* 2013)

Let $g, f \geq 0$ and suppose that f is bounded away from zero, that is,

$$0 < \delta \leq f \leq M < \infty$$

for some δ and M. Then there exists $R \geq M + 1$ and $C_1, C_2 > 0$ such that

$$H(g|f) \geq \begin{cases} C_1|g - f|^2, & \text{if } g \in [0, R] \\ C_2|g - f|^p, & \text{if } g \in (R, \infty) \end{cases}$$

Let $(X, \mathbb{X}, \mu)$ be a measure space.

Let $f_n, f : X \to [0, \infty)$ be measurable and such that f is bounded away from zero for a.e $x \in X$.

Let the set B_n be defined as

$$B_n = \{x \in X \mid 0 \leq f_n(x) \leq R\}$$

where R is as in the lemma.

In these conditions one deduces that

$$\int_X H(f_n|f)\, d\mu \geq C_1 \int_{B_n} |f_n - f|^2\, dx + C_2 \mu(B_n^c)$$

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Question : $\int_X H(f_n|f)\,d\mu \to 0 \quad \overset{?}{\Rightarrow} \quad \exists$ subsequence (f_{k_n}) that converges to f a.e.

This motivates the following definition:

Definition

A sequence (f_n) of measurable functions is said to α_p-converge to a measurable function f if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c) \to 0$ as $n \to \infty$ such that

$$\int_{B_n} |f_n - f|^p\,d\mu \to 0 \text{ as } n \to \infty.$$

Relative entropy : $H(g|f) = H(g) - H(f) - H'(f)(g - f)$

Question : $\int_X H(f_n|f)\,d\mu \to 0 \quad \overset{?}{\Rightarrow} \quad \exists$ subsequence (f_{k_n}) that converges to f a.e.

Assume that X is a bounded measurable subset of $\mathbb{R}^d$.

Theorem (A., J. Skrzeczkowski, A. E. Tzavaras, *in preparation*)

Let $p > 1$ and suppose that $H : \mathbb{R} \to [0, \infty)$ is continuously differentiable, strictly convex and satisfies

$$c|\lambda|^p - c \leq H(\lambda) \leq c|\lambda|^p + c$$

for some constant $c > 0$ and every $\lambda \in \mathbb{R}$. Let $u_n, u : X \to \mathbb{R}$ belong to $L^p(X)$. Then

$$\int_X H(u_n|u)\,dx \to 0 \qquad \text{if and only if} \qquad u_n \to u \text{ in } L^p(X)$$

Modes of convergence that are almost in L_p

Let $f_n, f : X \to \mathbb{R}$ be measurable. The sequence (f_n) is said to converge to f:

- in L_p if
$$\int_X |f_n - f|^p \, d\mu \to 0 \text{ as } n \to \infty$$
- in measure if
$$\forall \delta > 0 \; \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \delta\}) \to 0 \text{ as } n \to \infty$$
- almost uniformly if
$$\forall \delta > 0 \; \exists E_\delta \in \mathbb{X} \text{ with } \mu(E_\delta) < \delta \text{ such that } f_n \to f \text{ uniformly on } E_\delta^c$$
- almost everywhere if
$$\exists N \in \mathbb{X} \text{ with } \mu(N) = 0 \text{ such that } f_n \to f \text{ pointwise on } N^c$$

Definition (α_p)

A sequence (f_n) of measurable functions is said to ***α_p-converge*** *to a measurable function f if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c) \to 0$ as $n \to \infty$ such that*

$$\int_{B_n} |f_n - f|^p \, d\mu \to 0 \text{ as } n \to \infty$$

as $n \in \infty$.

Definition (almost in L_p)

A sequence (f_n) of measurable functions is said to **converge almost in L_p** to a measurable function f if for each $\delta > 0$ there exists a measurable set E_δ with $\mu(E_\delta) < \delta$ such that

$$\int_{E_\delta^c} |f_n - f|^p \, d\mu \to 0 \text{ as } n \to \infty.$$

It is always the case that the following relations hold

$$\begin{aligned} L_p\text{-convergence} &\Rightarrow \text{convergence almost in } L_p \\ &\Rightarrow \alpha_p\text{-convergence} \\ &\Rightarrow \text{convergence in measure} \end{aligned}$$

Interestingly enough, none of this modes of convergence are equivalent.

Moreover, if $\mu(X) < \infty$ then

$$\begin{aligned} \text{convergence a.e.} &\Rightarrow \text{convergence a.u.} \\ &\Rightarrow \text{convergence almost in } L_p \\ &\Rightarrow \alpha_p\text{-convergence} \end{aligned}$$

$$\begin{aligned} L_p\text{-convergence} &\Rightarrow \text{convergence almost in } L_p \\ &\Rightarrow \alpha_p\text{-convergence} \\ &\Rightarrow \text{convergence in measure} \end{aligned}$$

Example 1:

Let $(X, \mathbb{X}, \mu) = ([0,1], \mathcal{B}, \mathcal{L})$.

Let $f_n = n^{1/p} \chi_{[0,1/n]}$.

Then (f_n) converges to 0 almost in L_p (and hence α_p-converges as well) but it does not converge to 0 in L_p.

$$
\begin{aligned}
L_p\text{-convergence} &\Rightarrow \text{convergence almost in } L_p \\
&\Rightarrow \alpha_p\text{-convergence} \\
&\Rightarrow \text{convergence in measure}
\end{aligned}
$$

Example 2:

Let $(X, \mathbb{X}, \mu) = ([0, \infty), \mathcal{B}, \mathcal{L})$

Let $f_n = \frac{1}{n^{1/p}} \chi_{[0,n]}$.

Then (f_n) converges to 0 in measure but it does not α_p-converge to 0 (nor almost in L_p).

convergence almost in $L_p \Rightarrow \alpha_p$-convergence

Example 3:

Let $(X, \mathbb{X}, \mu) = ([0,1], \mathcal{B}, \mathcal{L})$

Consider the following sequence of measurable sets:

$$F_1 = [0,1]$$
$$F_2 = [0,1/2], \quad F_3 = [1/2,1]$$
$$F_4 = [0,1/3], \quad F_5 = [1/3,2/3], \quad F_6 = [2/3,1], \quad \text{and so on}$$

Let $f_n = \frac{1}{\mu(F_n)^{1/p}} \chi_{F_n}$.

Then (f_n) α_p-converges to 0, but it does not converge to 0 almost in L_p.

Convergence almost in L_p is naturally related to almost-L_p spaces.

A measurable function f is said to be almost in L_p if for each $\delta > 0$ there exists a measurable set E_δ with $\mu(E_\delta) < \delta$ such that

$$\int_{E_\delta^c} |f|^p \, d\mu < \infty$$

- Bravo, O. G., & Pérez, E. A. S. Optimal range theorems for operators with p-th power factorable adjoints. *Banach Journal of Mathematical Analysis* (2012)
- Calabuig, J. M., Bravo, O. G., Juan, M. A., & Pérez, E. A. S. Representation and factorization theorems for almost-L_p-spaces. *Indagationes Mathematicae* (2019)

- $f_n \xrightarrow{\alpha_p} f \quad \Leftrightarrow \quad \exists (B_n) \subseteq X$ with $\mu(B_n^c) \to 0$ such that

$$\int_{B_n} |f_n - f|^p \, d\mu \to 0$$

- $f_n \to f$ almost in $L_p \quad \Leftrightarrow \quad \forall \delta > 0 \; \exists E_\delta \in \mathbb{X}$ with $\mu(E_\delta) < \delta$ such that

$$\int_{E_\delta^c} |f_n - f|^p \, d\mu \to 0$$

Proposition

If (f_n) α_p-converges to f, then there exists a subsequence (f_{k_n}) that converges to f almost in L_p.

A sequence (f_n) of measurable functions is said to be:

- $\boldsymbol{\alpha_p}$**-Cauchy** if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c) \to 0$ as $n \to \infty$ such that

$$\int_{B_n \cap B_m} |f_n - f_m|^p \, d\mu \to 0 \text{ as } n, m \to \infty$$

- **Cauchy almost in** $\boldsymbol{L_p}$ if for each $\delta > 0$ there exists a measurable set E_δ with $\mu(E_\delta) < \delta$ such that

$$\int_{E_\delta^c} |f_n - f_m|^p \, d\mu \to 0 \text{ as } n, m \to \infty$$

Both notions of convergence are complete in the sense that Cauchy sequences converge (in the corresponding mode of convergence) to a limit.

Proposition

Let $(f_n), f$ *be measurable functions. Then*

$$f_n \xrightarrow{\alpha_p} f \quad \Leftrightarrow \quad \begin{cases} f_n \to f \text{ in measure,} \\ \exists \delta > 0 \ \displaystyle\int_{E_n^c(\delta)} |f_n - f|^p \, d\mu \to 0 \text{ as } n \to \infty, \end{cases}$$

where $E_n(\delta) = \{x \in X \mid |f_n(x) - f(x)| \geq \delta\}$.

If (f_n) converges to f in measure, then the following conditions are equivalent:

- $\exists \delta > 0 \ \int_{E_n^c(\delta)} |f_n - f|^p \, d\mu \to 0$ as $n \to \infty$
- $\forall \delta > 0 \ \int_{E_n^c(\delta)} |f_n - f|^p \, d\mu \to 0$ as $n \to \infty$

Vitali Convergence Theorem

Let $(f_n) \subseteq L_p$ and f be measurable. Then (f_n) converges to f in L_p if and only if

- (f_n) converges to f in measure,
- for every $\varepsilon > 0$ there exist $E_\varepsilon \in \mathbb{X}$ with $\mu(E_\varepsilon) < \infty$ and $\delta_\varepsilon > 0$ such that

$$\sup_n \int_{E_\varepsilon^c} |f_n|^p \, d\mu < \varepsilon^p$$

and

$$\forall F \in \mathbb{X} \qquad \mu(F) < \delta_\varepsilon \;\Rightarrow\; \sup_n \int_{E_\varepsilon \cap F} |f_n|^p \, d\mu < \varepsilon^p$$

Vitali Convergence Theorem for α_p-convergence and convergence almost in L_p

Let $(f_n) \subseteq L_p$ and f be measurable. Then (f_n) converges to f in L_p if and only if

- (f_n) α_p-converges or converges almost in L_p to f,
- for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\forall F \in \mathbb{X} \qquad \mu(F) < \delta_\varepsilon \;\Rightarrow\; \sup_n \int_F |f_n|^p \, d\mu < \varepsilon^p$$

Preservation of convergence under composition

Let $\mathfrak{m}$ be a notion of convergence for sequences of measurable functions.

A function $\varphi : \mathbb{R} \to \mathbb{R}$ is said to preserve $\mathfrak{m}$-convergence if given a measure space $(X, \mathbb{X}, \mu)$ and a sequence (f_n) of $\mathbb{X}$-measurable functions that $\mathfrak{m}$-converges to f, then the sequence $(\varphi(f_n))$ $\mathfrak{m}$-converges to $\varphi(f)$.

Theorem (Bartle, Joichi, *Proc. AMS* 1961)

- *φ preserves almost everywhere convergence if and only if φ is continuous,*
- *φ preserves uniform convergence, almost uniform convergence or convergence in measure if and only if φ is uniformly continuous.*

Theorem

A function $\varphi : \mathbb{R} \to \mathbb{R}$ preserves L_p-convergence, convergence almost in L_p or α_p-convergence if and only if φ is Lipschitz continuous.

Lemma

A function $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous if and only if

$$\exists \delta > 0 \quad \exists K > 0 \quad \forall a, b \in \mathbb{R} \quad |a - b| < \delta \Rightarrow |\varphi(a) - \varphi(b)| \leq K|a - b|$$

Proof of Characterization Theorem

($\Leftarrow$) Suppose that φ is Lipschitz continuous and that $f_n \to f$ in L_p. Then, there exists $K > 0$ such that

$$\forall a, b \in \mathbb{R} \quad |\varphi(a) - \varphi(b)| \leq K|a - b|$$

and

$$\int_X |f_n - f|^p \, d\mu \to 0 \text{ as } n \to \infty$$

Proof of Characterization Theorem

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$$\forall a, b \in \mathbb{R} \quad |\varphi(a) - \varphi(b)| \leq K|a - b|$$

and

$$\int_X |f_n - f|^p \, d\mu \to 0 \text{ as } n \to \infty$$

Thus,

$$\int_X |\varphi(f_n) - \varphi(f)|^p \, d\mu \leq K^p \int_X |f_n - f|^p \, d\mu \to 0 \text{ as } n \to \infty,$$

that is,

$$\varphi(f_n) \to \varphi(f) \text{ in } L_p.$$

In a similar fashion, if $f_n \to f$ almost in L_p or $f_n \xrightarrow{\alpha_p} f$ then $\varphi(f_n) \to \varphi(f)$ almost in L_p or $\varphi(f_n) \xrightarrow{\alpha_p} \varphi(f)$, respectively.

$(\Rightarrow)$

Suppose that φ is not Lipschitz continuous. According to the lemma, choose sequences $(a_n), (b_n) \subseteq \mathbb{R}$ such that, for each $n \in \mathbb{N}$,

$$0 < |a_n - b_n| < \frac{1}{n^{1/p}}, \quad |\varphi(a_n) - \varphi(b_n)| > n^{1/p}|a_n - b_n|.$$

($\Rightarrow$)
Suppose that φ is not Lipschitz continuous. According to the lemma, choose sequences $(a_n), (b_n) \subseteq \mathbb{R}$ such that, for each $n \in \mathbb{N}$,

$$0 < |a_n - b_n| < \frac{1}{n^{1/p}}, \quad |\varphi(a_n) - \varphi(b_n)| > n^{1/p}|a_n - b_n|.$$

Let $(X, \mathbb{X}, \mu) = ([0, \infty), \mathcal{B}, \mathcal{L})$, define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} b_1, & \text{if } 0 \leq x < \frac{1}{|a_1 - b_1|^p} \\ b_n, & \text{if } \sum_{k=1}^{n-1} \frac{1}{|a_k - b_k|^p} + \frac{n-1}{n|a_n - b_n|^p} \leq x < \sum_{k=1}^{n} \frac{1}{|a_k - b_k|^p}, \quad n \in \mathbb{N} \setminus \{1\} \\ 0, & \text{otherwise} \end{cases}$$

and, for each $n \in \mathbb{N} \setminus \{1\}$, let $f_n : X \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} a_n, & \text{if } \sum_{k=1}^{n-1} \frac{1}{|a_k - b_k|^p} + \frac{n-1}{n|a_n - b_n|^p} \leq x < \sum_{k=1}^{n} \frac{1}{|a_k - b_k|^p} \\ f(x), & \text{otherwise} \end{cases}$$

It holds that

$$\int_X |f_n - f|^p \, d\mu = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

that is, (f_n) converges to f in L_p. Hence $f_n \to f$ almost in L_p and $f_n \xrightarrow{\alpha_p} f$.

It remains to be shown that $(\varphi(f_n))$ does not α_p-converge to $\varphi(f)$ (and hence it does not converge neither almost in L_p nor in L_p).

To that end, let (B_n) be any sequence of measurable sets such that $\mu(B_n^c) \to 0$ as $n \to \infty$, set

$$I_n = \left[\sum_{k=1}^{n-1} \frac{1}{|a_k - b_k|^p} + \frac{n-1}{n|a_n - b_n|^p}, \sum_{k=1}^{n} \frac{1}{|a_k - b_k|^p}\right),$$

and notice that

$$\begin{aligned}\int_{B_n} |\varphi(f_n) - \varphi(f)|^p \, d\mu &= \int_{B_n \cap I_n} |\varphi(a_n) - \varphi(b_n)|^p \, d\mu \\ &> n|a_n - b_n|^p \mu(B_n \cap I_n) \\ &= n|a_n - b_n|^p \big(\mu(I_n) - \mu(B_n^c \cap I_n)\big) \\ &= 1 - n|a_n - b_n|^p \mu(B_n^c \cap I_n).\end{aligned}$$

Now let $N \in \mathbb{N}$ be such that $\mu(B_n^c) < 1/2$ whenever $n \geq N$. Thus, for $n \geq N$,

$$\begin{aligned}\int_{B_n} |\varphi(f_n) - \varphi(f)|^p \, d\mu &> 1 - n|a_n - b_n|^p \mu(B_n^c \cap I_n) \\ &> 1 - \mu(B_n^c) \\ &> 1/2\end{aligned}$$

therefore $(\varphi(f_n))$ does not α_p-converge to $\varphi(f)$ and the proof is complete.

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A., J. Paulos, *A mode of convergence arising in diffusive relaxation*, Q. J. Math 75(1), 143–159, 2024.

Thank you