On a triangle modification of the Niemytzki plane

Wojciech Bielas

University of Silesia in Katowice

June 18, 2024

The Niemytzki plane

Let us recall that the Niemytzki plane is a topological space

$$N = \{(x, y) \in \mathbb{R}^2 \colon y \geqslant 0\},$$

where

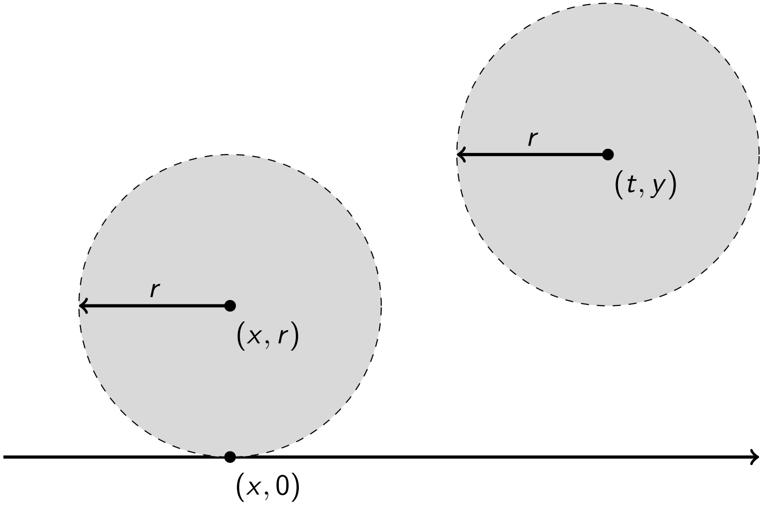
- neighbourhoods of a point $(x, y) \in N$, with $y > 0$, are of the form

$$D(x, y, r) = N \cap \{(z, t) \in \mathbb{R}^2 \colon (x - z)^2 + (y - t)^2 < r^2\},$$

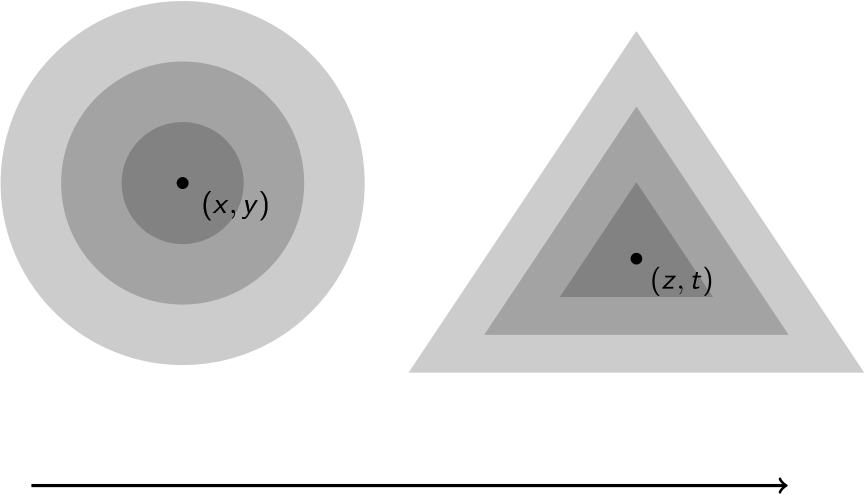
- neighbourhoods of a point $(x, 0) \in N$ are of the form

$$D_0(x, r) = \{(x, 0)\} \cup D(x, r, r).$$

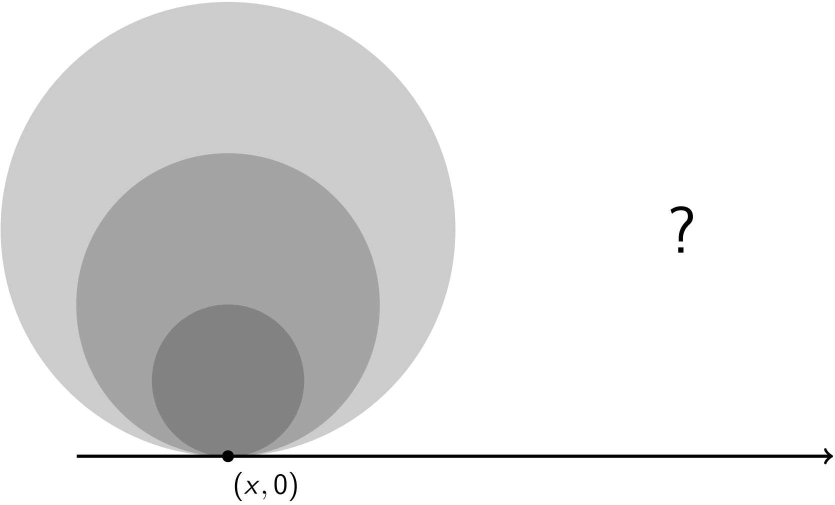
The Niemytzki plane



Base at a point (x, y) with $y > 0$

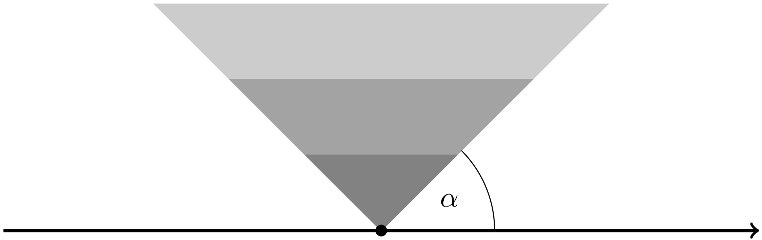


Base at a point $(x, 0)$



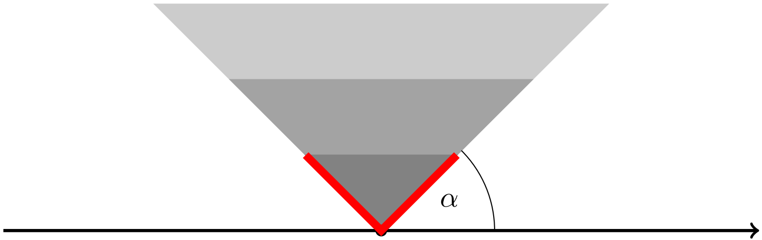
Triangles with a fixed angle α

This topology is not regular: the boundary of a smaller triangle is not contained in a bigger triangle.



Triangles with a fixed angle α

This topology is not regular: the boundary of a smaller triangle is not contained in a bigger triangle.

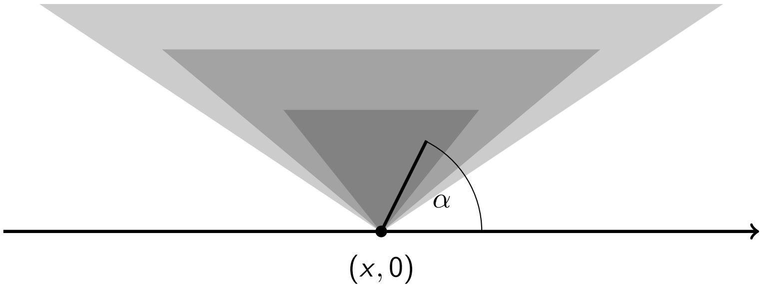


Triangles with angles $< \alpha$

We define

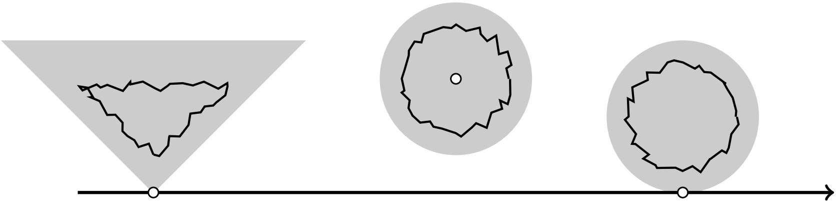
$$T(\beta, x, n) = \{(z, y) \in N \colon |(\tan\beta)(z - x)| < y < \tfrac{1}{n}\} \cup \{(x, 0)\},$$

where $\beta < \alpha$ and α is fixed.



Nonhomogeneity of the Niemytzki plane

- Let us observe that if $U \subseteq N$ is a neighbourhood of (x, y) with $y > 0$, then $U \setminus \{(x, y)\}$ contains paths which cannot be contracted to a point.
- This does not hold for points $(x, 0)$.
- The triangle modification N_T has the same property.
- If $f : N \to N_T$ is a homeomorphism, then $f(x, 0) = (y, 0)$ for a unique y and this defines a function $g : \mathbb{R} \to \mathbb{R}$, $f(x, 0) = (g(x), 0)$.

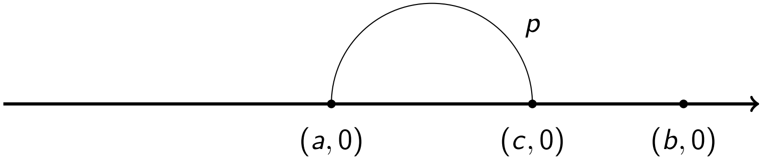


Restriction of homeomorphisms to the real line

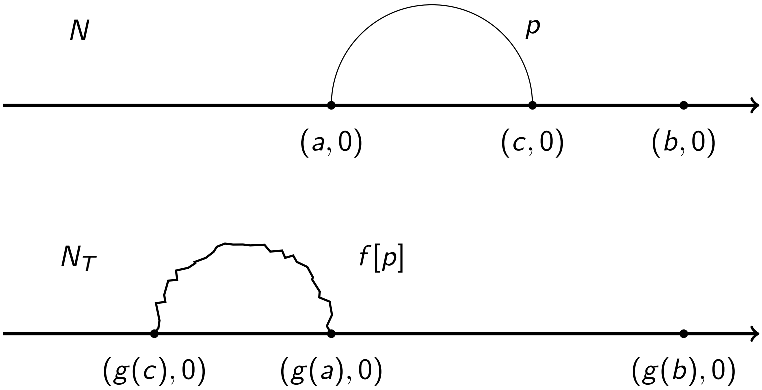
Proposition

If $f\colon N \to N_T$ is a homeomorphism and $g\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(x,0) = (g(x),0)$, then there exists an open interval $(a,c) \neq \emptyset$ such that $g[(a,c)]$ is also an open interval.

Fix $a, b \in \mathbb{R}$ such that $a < b$ and $g(a) < g(b)$. Assume that there is $c \in (a,b)$ such that $g(c) < g(a)$. Let $p \subseteq N$ be an arc connecting $(a,0)$ and $(c,0)$.



Fix $a, b \in \mathbb{R}$ such that $a < b$ and $g(a) < g(b)$. Assume that there is $c \in (a, b)$ such that $g(c) < g(a)$. Let $p \subseteq N$ be an arc connecting $(a, 0)$ and $(c, 0)$.



Points from $(a, c) \times \{0\}$ and the point $(b, 0)$ cannot be connected by an arc disjoint from p. The same can be said about points from $(g(a), g(c)) \times \{0\}$ and the point $(g(b), 0)$.

Proposition

If $f\colon N \to N_T$ is a homeomorphism and $g\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(x,0) = (g(x),0)$, then there exists an open interval $(a,c) \neq \emptyset$ such that $g[(a,c)]$ is an open interval and $g|_{(a,c)}$ is monotone.

- Fix $a < d < e < c$ and suppose that $g(a) < g(e) < g(d) < g(c)$.
- Let p be an arc from $(a,0)$ to $(d,0)$ and q be an arc from $(e,0)$ to $(c,0)$ such that $p \cap q = \emptyset$.
- Then $f[p], f[q]$ are disjoint arcs and $(g(a),0), (g(d),0) \in f[p]$, $(g(e),0), (g(c),0) \in f[q]$; a contradiction.

- We start once again with a homeomorphism $f\colon N \to N_T$, the function g such that $f(x,0) = (g(x),0)$ and an interval $(A,B) \subseteq \mathbb{R}$ such that $g|_{(A,B)}$ is increasing.
- For every $x \in \mathbb{R}$ there exists n_x such that

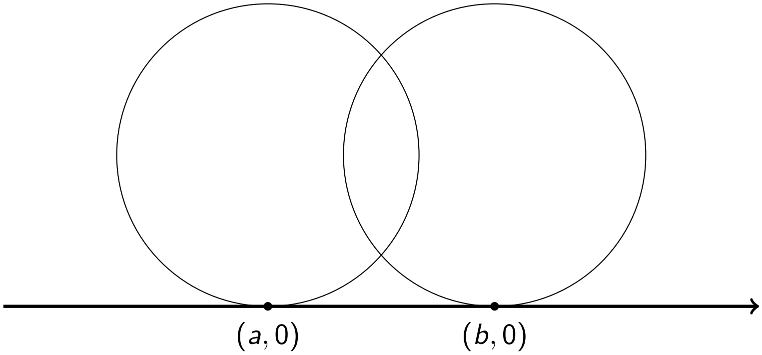
$$f[D_0(x,n_x)] \subseteq T(\tfrac{\alpha}{2}, g(x), 1).$$

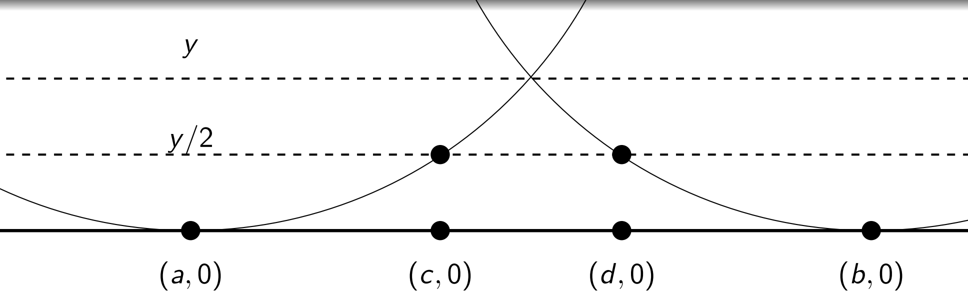
- The Baire category theorem implies that there exists a nonempty interval $(a,b) \subseteq (A,B)$ and $n \geqslant 1$ such that the set

$$G = \{x \in \mathbb{R}\colon\, n_x = n\}$$

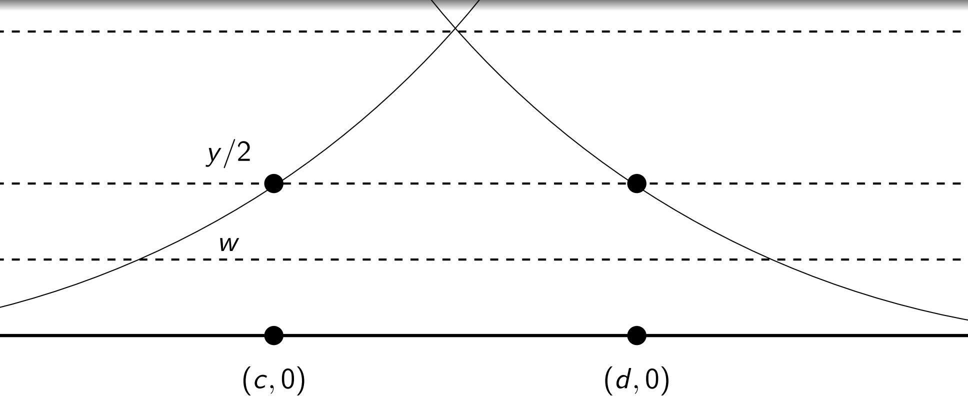
is dense in (a,b).

- Considering a smaller interval (a, b), we can assume that boundaries of $D_0(a, n), D_0(b, n)$ intersect in two points: $(\frac{a+b}{2}, y), (\frac{a+b}{2}, z)$, where $y < z$.

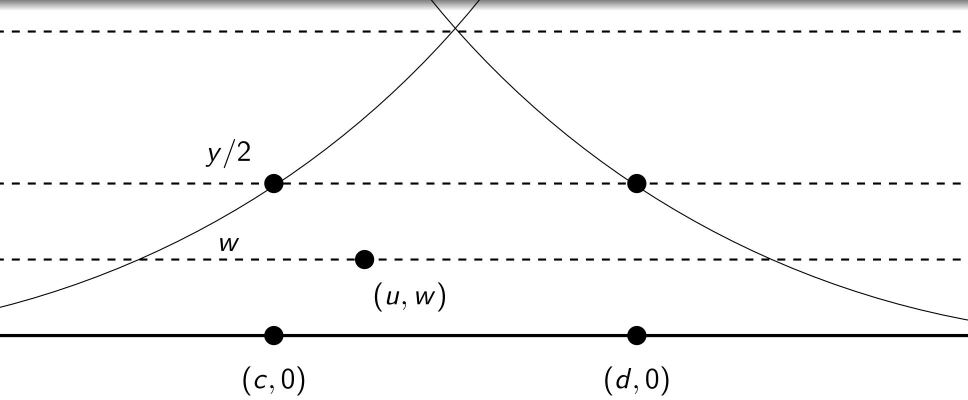




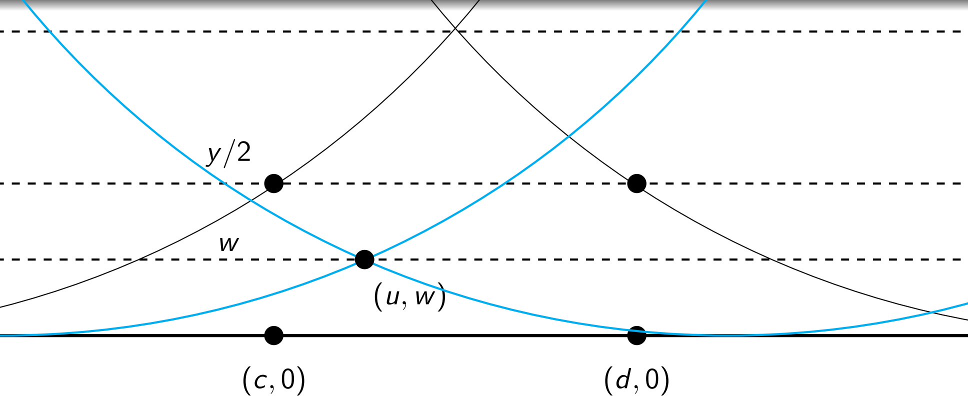
- There exist unique $a < c < \frac{a+b}{2} < d < b$ such that points $(c, \frac{y}{2}), (d, \frac{y}{2})$ belong to the boundary of $D_0(a, n) \cup D_0(b, n)$.



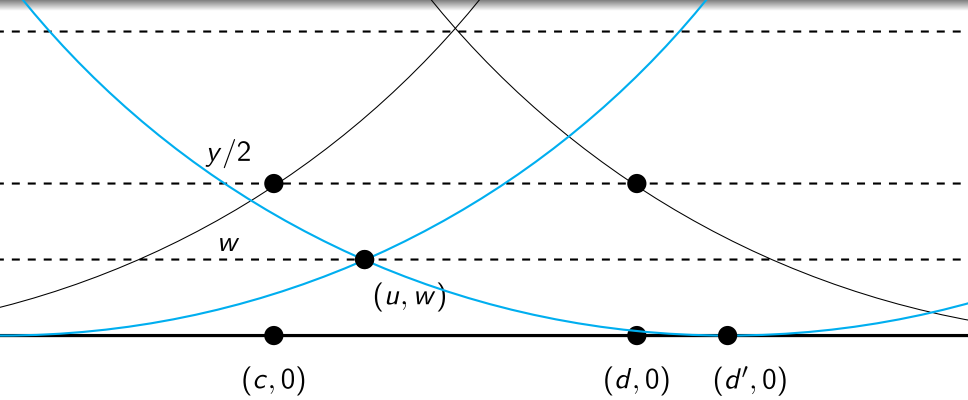
- For every $u \in (c, d)$ and $w \in (0, \frac{y}{2})$ there exist $a < c' < c$ and $d < d' < b$ such that the point (u, w) belong to boundaries of $D_0(c', n), D_0(d', n)$.



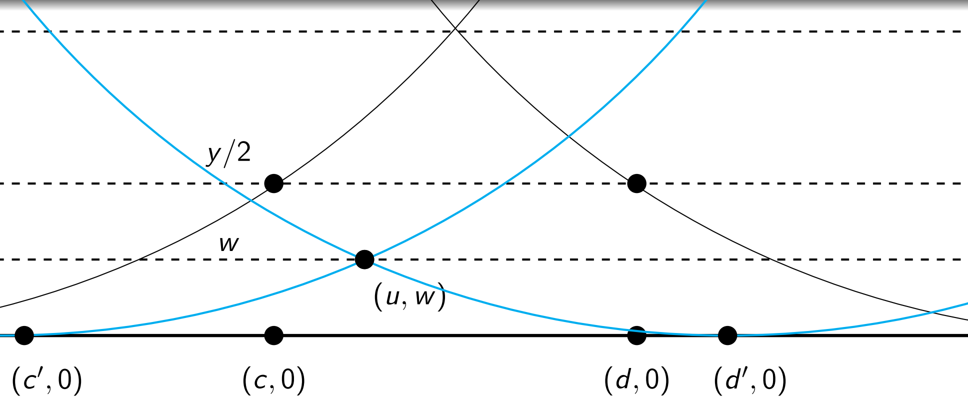
- For every $u \in (c, d)$ and $w \in (0, \frac{y}{2})$ there exist $a < c' < c$ and $d < d' < b$ such that the point (u, w) belong to boundaries of $D_0(c', n), D_0(d', n)$.



- For every $u \in (c, d)$ and $w \in (0, \frac{y}{2})$ there exist $a < c' < c$ and $d < d' < b$ such that the point (u, w) belong to boundaries of $D_0(c', n), D_0(d', n)$.

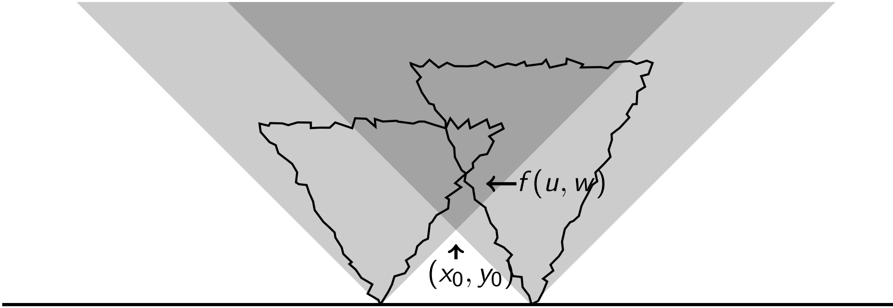


- For every $u \in (c, d)$ and $w \in (0, \frac{y}{2})$ there exist $a < c' < c$ and $d < d' < b$ such that the point (u, w) belong to boundaries of $D_0(c', n), D_0(d', n)$.



- For every $u \in (c, d)$ and $w \in (0, \frac{y}{2})$ there exist $a < c' < c$ and $d < d' < b$ such that the point (u, w) belong to boundaries of $D_0(c', n), D_0(d', n)$.

- Assume that $c' \in G$ and let $(x_m, y_m) \in D_0(c', n)$ be such that $(x_m, y_m) \to (u, w)$.
- Then $f(x_m, y_m) \in f[D_0(c', n)] \subseteq T(\frac{\alpha}{2}, g(c'), 1)$ and $f(u, w) \in \mathrm{cl}\, T(\frac{\alpha}{2}, g(c'), 1)$.
- Similarly, if $d' \in G$, then $f(u, w) \in \mathrm{cl}\, T(\frac{\alpha}{2}, g(d'), 1)$.
- If $f(u, w) = (f_1(u, w), f_2(u, w))$, then $y_0 \leqslant f_2(u, w)$.



- We have $x_0 = \frac{g(c')+g(d')}{2}$ and

$$y_0 = (\tan\tfrac{\alpha}{2})(x_0 - g(c')) = (\tan\tfrac{\alpha}{2})(\tfrac{g(c')+g(d')}{2} - g(c')),$$

hence

$$\left(\tan\frac{\alpha}{2}\right)\frac{g(u)-g(c')}{2} \leqslant \left(\tan\frac{\alpha}{2}\right)\frac{g(d')-g(c')}{2} \leqslant f_2(u,w).$$

- Point (u, w) belongs to the boundary of $D_0(c', n)$, hence

$$(u-c')^2 + (w - \tfrac{1}{n})^2 = (\tfrac{1}{n})^2,$$

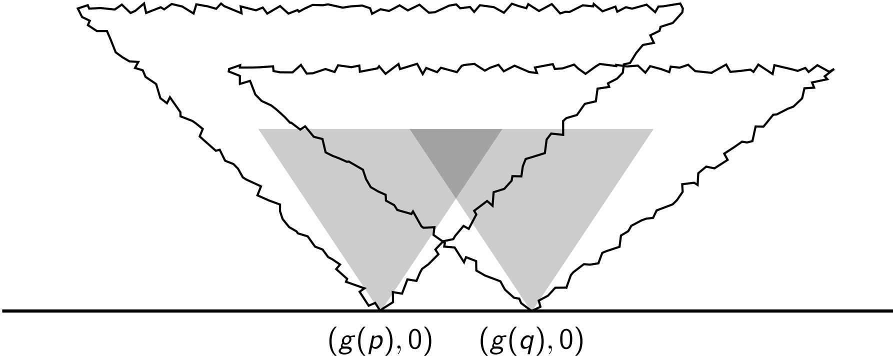
$$c' = u - \sqrt{\frac{2}{n}w - w^2},$$

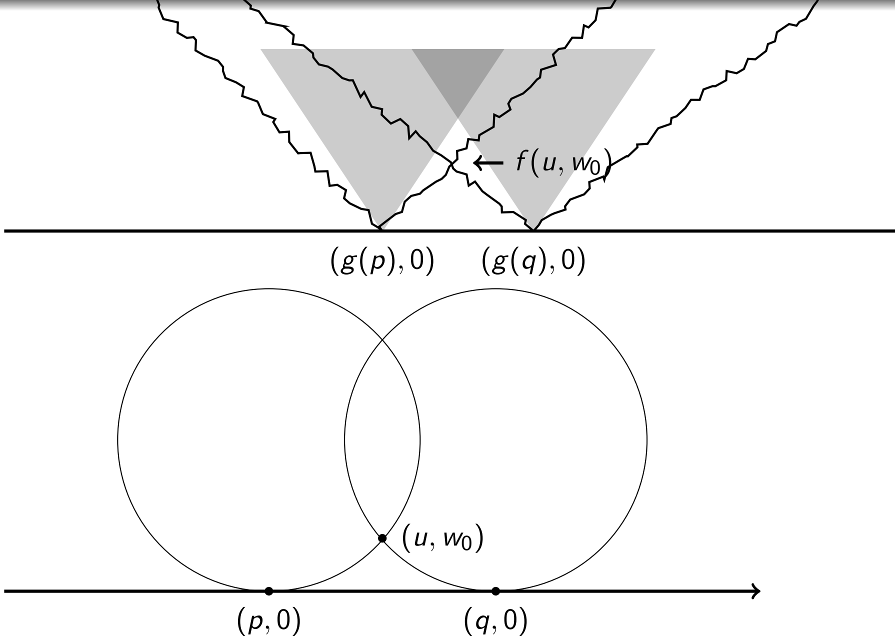
and

$$(*) \quad \left(\tan\frac{\alpha}{2}\right)\frac{g(u) - g\left(u - \sqrt{\frac{2}{n}w - w^2}\right)}{2} \leqslant f_2(u,w).$$

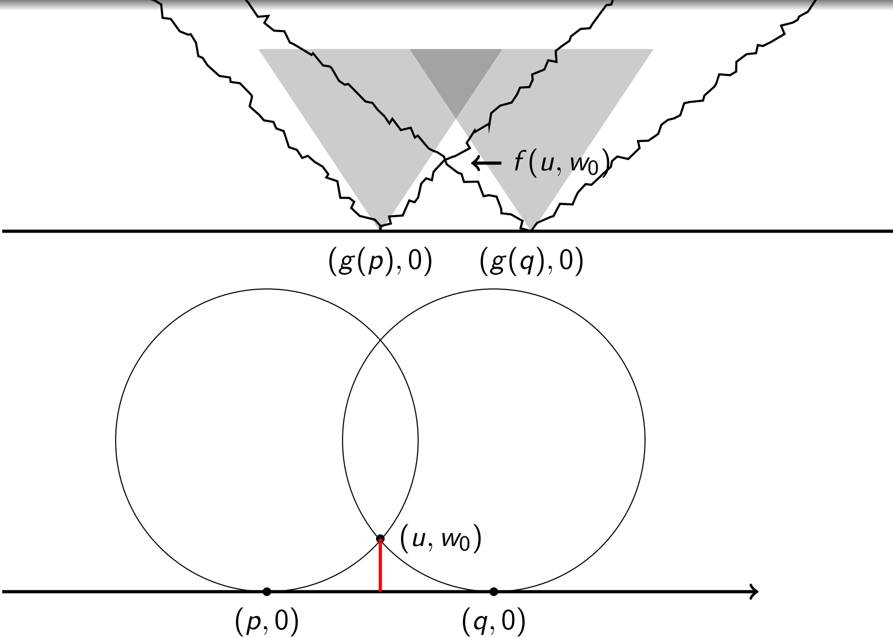
- There exists m such that $\frac{2}{m} < \frac{y}{2}$ and
$$\sqrt{\frac{n}{m}} < \frac{\tan\frac{\alpha}{2}}{\tan\alpha}.$$
- For every $u \in (c, d)$ there exists k_u and $\beta_u \in (\frac{\alpha}{2}, \alpha) \cap \mathbb{Q}$ such that
$$T(\beta_u, g(u), k_u) \subseteq f[D_0(u, m)].$$
- The Baire category theorem implies that there exists an interval $(s, t) \subseteq (c, d)$, an angle $\beta \in (\frac{\alpha}{2}, \alpha)$ and k such that the set
$$H = \{u \in (s, t) \colon (\beta_u, k_u) = (\beta, k)\}$$
is dense in (s, t).

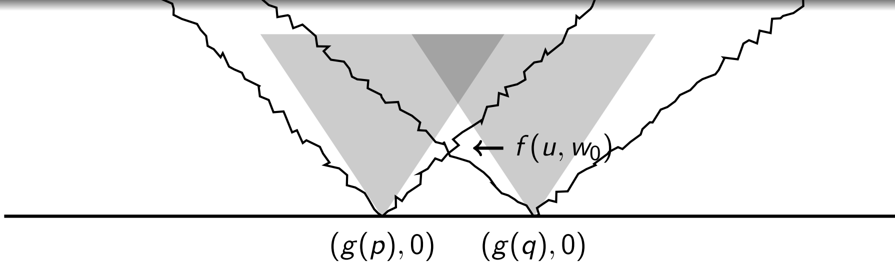
- Fix $s < p < q < t$ such that $p, q \in H$ and the intersection of triangles $T(\beta, g(p), k)$, $T(\beta, g(q), k)$ has nonempty interior.
- Then $f[D_0(p,m)] \cap f[D_0(q,m)]$ has nonempty interior and the intersection of boundaries of $D_0(p,m), D_0(q,m)$ has two points $(u, w_0), (u, w_1)$, where $w_0 < w_1$ and $u = \frac{p+q}{2}$.





$\leftarrow f(u, w_0)$
$(g(p), 0)$
$(g(q), 0)$
(u, w_0)
$(p, 0)$
$(q, 0)$


$\leftarrow f(u, w_0)$
$(g(p), 0)$
$(g(q), 0)$
(u, w_0)
$(p, 0)$
$(q, 0)$



- We obtain the inequality

$$f(u, w_0) \leqslant (\tan \beta) \frac{g(q) - g(p)}{2}.$$

- Point (u, w_0) belongs to the boundary of $D_0(p, m)$, hence

$$(u - p)^2 + (w_0 - \tfrac{1}{m})^2 = (\tfrac{1}{m})^2,$$

and $p = u - \sqrt{\frac{2}{m} w_0 - w_0^2}$.

Since $g(u) < g(q)$, we obtain

$$f_2(u, w_0) \leqslant (\tan\beta)\frac{g(u) - g(u - \sqrt{\frac{2}{m}w_0 - w_0^2})}{2}.$$

and, together with $(*)$,

$$\left(\tan\frac{\alpha}{2}\right)\frac{g(u) - g(u - \sqrt{\frac{2}{n}w_0 - w_0^2})}{2} \leqslant$$

$$(\tan\beta)\frac{g(u) - g(u - \sqrt{\frac{2}{m}w_0 - w_0^2})}{2}.$$

- Let us denote $t_i = \sqrt{\frac{2}{i} w_0 - w_0^2}$.
- Then

$$\frac{\tan\frac{\alpha}{2}}{\tan\alpha} < \frac{\tan\frac{\alpha}{2}}{\tan\beta} \leqslant \frac{g(u) - g(u - t_m)}{g(u) - g(u - t_n)} = \\ = \frac{g(u) - g(u - t_m)}{t_m} \cdot \frac{t_n}{g(u) - g(u - t_n)} \cdot \frac{t_m}{t_n}.$$

- If g is differentiable at the point u and $w_0 \to 0$, then the right hand side tends to $\sqrt{\frac{n}{m}}$; a contradiction.