Combinatorial Banach spaces

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This represents joint works with Barnabas Farkas, Sebastian Jachimek, Jordi Lopez-Abad, Anna Pelczar-Barwacz.

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This is a slang coming from set theory (analytic P-ideals).

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Convention

$X_{\mathcal{F}} = Exh(\mathcal{F})$.

The canonical example

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So, if $\mathcal{F}$ is compact, then $X_{\mathcal{F}}$ is c_0-saturated.

Each countable compact space is scattered (i.e. it does not contain a Cantor set). One can analyze $X_{\mathcal{F}}$ for compact families in terms of Cantor-Bendixson rank of $\mathcal{F}$ (higher order Schreier spaces).

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[Contemplate $X_{\mathcal{A}}$ and $X_{\mathcal{C}}$.]

Farah space

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In general:

Schur property $\implies$ ℓ_1-saturation $\implies$ no copies of c_0

Schur property vs ℓ_1-saturation

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Then $X_{\mathcal{F}}$ does not have the Schur property but it is ℓ_1-saturated.

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Even more, Rosenthal proved that $X_{\mathcal{C}}$ is a universal space for all Banach spaces with unconditional basis!

So, combinatorial spaces may be quite reach in terms of subspaces.

A 'visible' copy of ℓ_2

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$$\|y\|_{\mathcal{F}} = \|\sum_n y(n) z_n\|_{\mathcal{F}} = \sup_{F \in \mathcal{F}} \sum_n \frac{|F \cap I_n|}{|I_n|} |y(n)| = \sup_{F \in \mathcal{F}} \langle \frac{|F \cap I_n|}{|I_n|}, y \rangle.$$

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In our case, let

$$T(x)(k) = \sup_{i \in I_n} \frac{x(i)}{|I_n|}$$

if $k \in I_n$.

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By a theorem due to Pełczyński $X_{\mathcal{F}}$ is isomorphic to so called Pełczyński space.

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Theorem No, $X_{\mathcal{F}}$ is not. In $X_{\mathcal{F}}$ the base is not universal (contrary to the case of the Pełczyński space).

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Theorem (Bang, Odell, 1989)

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Conversely, each reflexive family is of the form $Hom_1(c)$ for some coloring.

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Does not work either. And I don't know if anything reasonable may work.

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It may explain the similarities between $X^{\mathcal{F}}$ and $X^*_{\mathcal{F}}$.

Thanks.