Composition operators in spaces of sequences of bounded variation

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Space $bv_p(E)$

By $bv_p(E)$ for $p \geq 1$ (E a normed space), we denote the space of all sequences $x\colon \mathbb{N} \to E$ such that $\sum_{n=1}^{\infty} \|x(n+1) - x(n)\|^p < +\infty$.

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$$bv_1(E) \subset l^\infty(E), \quad bv_p(E) \not\subset l^\infty(E) \quad l^\infty(E) \not\subset bv_p(E)$$

Composition operator

Let E be a normed space and let Ω be a non-empty set. For given map $f\colon E \to E$ we define the *composition operator* C_f, generated by f, as

$$C_f(x)(t) = f(x(t)), \qquad \text{for } t \in \Omega,$$

where $x : \Omega \to E$.

Acting conditions with target space $bv_p(E)$

Theorem

Let $p, q \in [1, +\infty)$.

1. *If C_f maps $X \in \{bv_p(E), c(E), l^\infty(E)\}$ into $bv_q(E)$, then f is continuous on E.*

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1. *If* C_f *maps* $X \in \{bv_p(E), c(E), l^\infty(E)\}$ *into* $bv_q(E)$*, then* f *is continuous on* E.
2. *If* C_f *maps* $Y \in \{l^p(E), c_0(E)\}$ *into* $bv_q(E)$*, then* f *is continuous at* 0.

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Theorem

Let $p \in [1, +\infty)$. The composition operator C_f maps $c_0(E)$ into $bv_p(E)$ if and only if its generator f is locally constant at 0, that is, there exists a number $\delta > 0$ such that $f|_{\bar{B}_E(0,\delta)}$ is constant.

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Theorem

Let $p \in [1, +\infty)$ and let $X \in \{c(E), l^\infty(E)\}$. The composition operator C_f maps X into $bv_p(E)$ if and only if f is a constant function.

Acting conditions with target space $bv_p(E)$

Theorem

For any $p, q \in [1, +\infty)$ the following conditions are equivalent:

1. *the composition operator C_f maps $l^p(E)$ into $bv_q(E)$,*

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1. *the composition operator C_f maps $l^p(E)$ into $bv_q(E)$,*
2. *there exist $\delta > 0$ and $L \geq 0$ such that* $\|f(u) - f(w)\| \leq L\|u\|^{\frac{p}{q}} + L\|w\|^{\frac{p}{q}}$ *for all $u, w \in \bar{B}_E(0, \delta)$,*

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3. *there exist $\delta > 0$ and $L \geq 0$ such that $\|f(u) - f(0)\| \leq L \|u\|^{\frac{p}{q}}$ for every $u \in \bar{B}_E(0, \delta)$.*

Acting conditions between bv_p-spaces

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Corollary

Let $1 \leq q < p < +\infty$. The composition operator C_f acts between $bv_p(E)$ and $bv_q(E)$ if and only if f is a constant map.

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Theorem

Let $p \in [1, +\infty)$ and let E be a Banach space. The composition operator C_f maps $bv_1(E)$ into $bv_p(E)$ if and only if f is Hölder continuous on compact subsets of E with exponent $\frac{1}{p}$.

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Example

Let $a := (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots)$ and for $u \in c_{00}(\mathbb{R})$ let $f(u) = \frac{u}{\|u-a\|_\infty}$. f is Lipschitz continuous on compact subsets of $c_{00}(\mathbb{R})$. However, it does not generate a composition operator acting between $bv_1(c_{00}(\mathbb{R}))$ and $bv_1(c_{00}(\mathbb{R}))$. Define $x\colon \mathbb{N} \to c_{00}(\mathbb{R})$ by $x(n) := P_n(a)$. Since $\|x(n+1) - x(n)\|_\infty = (n+1)^{-2}$ for $n \in \mathbb{N}$, we see that $x \in bv_1(c_{00}(\mathbb{R}))$. But,

$$\sum_{n=1}^{\infty} \|f(x(n+1)) - f(x(n))\|_\infty = +\infty.$$

Hence, $C_f(x) \notin bv_1(c_{00}(\mathbb{R}))$.

Acting conditions between bv_p-spaces

Definition

A map $f : E \to E$ is called locally Hölder continuous in the stronger sense with exponent α, if there exist $\delta > 0$ and $L \geq 0$ such that $\|f(u) - f(w)\| \leq L \|u - w\|^\alpha$ for all $u, w \in E$ with $\|u - w\| \leq \delta$.

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Theorem

Let $1 < p \leq q < +\infty$. The composition operator C_f maps $bv_p(E)$ into $bv_q(E)$ if and only if f is locally Hölder continuous in the stronger sense with exponent $\frac{p}{q}$.

Composition operators acting from bv_p-spaces

Theorem

Let $p, q \in [1, +\infty)$ and let $X \in \{c_0(E), l^q(E)\}$. The composition operator C_f maps $bv_p(E)$ into X if and only if its generator f is the zero function.

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Theorem

Let E be a Banach space. The composition operator C_f maps $bv_1(E)$ into $l^\infty(E)$ if and only if f is bounded on compact subsets of E.

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Theorem

Let $p \in (1, +\infty)$. Then, the following conditions are equivalent:

1. *the composition operator C_f maps $bv_p(E)$ into $l^\infty(E)$,*
2. *f is bounded on E,*
3. *f is bounded on countable subsets of E.*

Composition operators acting from bv_p-spaces to $c(E)$

Theorem

Let E be a Banach space. Then, the following conditions are equivalent:

1. *the composition operator C_f maps $bv_1(E)$ into $c(E)$,*
2. *f is continuous on E,*
3. *f is continuous on bounded subsets of E,*
4. *f is continuous on compact subsets of E,*
5. *f is continuous on countable subsets of E.*

Composition operators acting from bv_p-spaces to $c(E)$

For $p > 1$ (Lipschitz) continuous mappings, in general, do not generate composition operators that act from $bv_p(E)$ into $c(E)$. To see this it suffices to consider the identity mapping on E, that is, $f\colon E \to E$ given by $f(u) = u$. It generates the identity composition operator C_f that maps $bv_p(E)$ onto $bv_p(E)$. But $bv_p(E) \not\subseteq c(E)$. And so, $C_f(bv_p(E)) \not\subseteq c(E)$.

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Theorem

Let $p \in (1, +\infty)$. The composition operator C_f maps $bv_p(E)$ into $c(E)$ if and only if f is a constant mapping.

Local boundedness

Theorem

Let $p \in [1, +\infty)$ and assume that the composition operator C_f maps $c_0(E)$ into $bv_p(E)$. Then, the following conditions are equivalent:

1. *C_f is bounded,*
2. *C_f is locally bounded,*
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Let $p, q \in [1, +\infty)$ and assume that the composition operator C_f maps $l^p(E)$ into $bv_q(E)$. Then, C_f is locally bounded if and only if f is.

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Theorem

Let $1 < p \leq q < +\infty$. Moreover, assume that the composition operator C_f maps $bv_p(E)$ into $bv_q(E)$. Then, C_f is locally bounded.

Local boundedness

Example

Let $f\colon l^2(\mathbb{R}) \to l^2(\mathbb{R})$ be defined by $f(u) = (\sin\psi(u), 0, \ldots)$, where $\psi(u) = \sup\{(2\pi+1)n|u(n)| - n;\ n \in \mathbb{N}\}$. f is Lipschitz continuous on compact subsets of $l^2(\mathbb{R})$, and so $C_f : bv_1(l^2(\mathbb{R})) \to bv_1(l^2(\mathbb{R}))$. For each $n \in \mathbb{N}$ set

$$x_n := \big(e_n, (1-n^{-2})e_n, e_n, (1-n^{-2})e_n, \ldots, e_n, (1-n^{-2})e_n, 0, 0, 0, \ldots\big),$$

where the last non-zero element appears at the $2n^2$-th position. Then, $x_n \in bv_1(l^2(\mathbb{R}))$ and $\|x_n\|_{bv_1} \leq 4$ for $n \in \mathbb{N}$. Since $f(e_n) = 0$ and $f\big((1-n^{-2})e_n\big) = (-\sin[(2\pi+1)n^{-1}], 0, 0, \ldots)$, for $n \geq 5$ we thus have

$$\|C_f(x_n)\|_{bv_1} \geq n^2 \left\| f(e_n) - f\big((1-\tfrac{1}{n^2})e_n\big)\right\|_{l^2} \geq 4n.$$

This means that C_f is not locally bounded.

Local boundedness

Theorem

Let $p \in [1, +\infty)$ *and let* E *be a Banach space. Moreover, assume that the composition operator* C_f *maps* $bv_1(E)$ *into* $bv_p(E)$*. Then,* C_f *is locally bounded if and only if* f *is Hölder continuous on bounded subsets of* E *with exponent* $\frac{1}{p}$*.*

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Let $p \in [1, +\infty)$ and let E be a Banach space. Moreover, assume that the composition operator C_f maps $bv_1(E)$ into $bv_p(E)$. Then, C_f is locally bounded if and only if f is Hölder continuous on bounded subsets of E with exponent $\frac{1}{p}$.

Theorem

Let E be a Banach space and let $X \in \{c(E), l^{\infty}(E)\}$. Moreover, assume that the composition operator C_f maps $bv_1(E)$ into X. Then, C_f is locally bounded if and only if f is.

Boundedness

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Thank you for your attention!