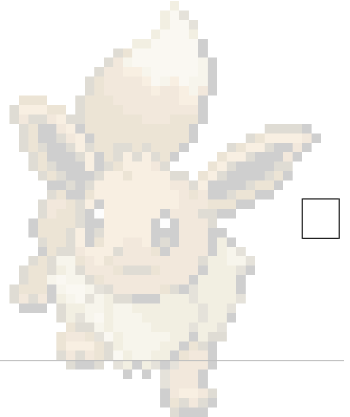
## Alan Chang (WashU): Prescribed projections and efficient coverings by curves in the plane

**Theorem 1** (Existence of Kakeya sets)
$\exists$ a set $K \subset \mathbb{R}^2$ which is a *union of lines* s.t. (1) $K$ contains a line in every direction and (2) $\mathcal{L}^2(K) = 0$.

**Theorem 2** (Dual formulation, a set with one large projection and many small projections)
$\exists$ a set $E \subset \mathbb{R}^2$ s.t. (1) $\operatorname{proj}_0 E \supset [0,1]$ and (2) for a.e. $\theta \in [0,\pi)$, $\mathcal{L}^1(\operatorname{proj}_\theta E) = 0$.



*Proof of Theorem 1 from Theorem 2.* Let $K = \displaystyle\bigcup_{(a,b)\in E} \{(x,y) \in \mathbb{R}^2 : y = ax + b\}$. $\square$

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**Theorem 3** (Davies's efficient covering theorem)
$\forall$ (measurable) $A \subset \mathbb{R}^2$, $\exists$ a set $K \subset \mathbb{R}^2$ which is a *union of lines* s.t. (1) $K \supset A$ and (2) $\mathcal{L}^2(K \setminus A) = 0$.

**Theorem 4** (Falconer's digital sundial theorem, a.k.a. Falconer's prescribed projection theorem)
Let $(A_\theta)_{\theta\in[0,\pi)}$ be a collection of subsets of $\mathbb{R}$ (such that $\bigcup_{\theta\in[0,\pi)}(\{\theta\} \times A_\theta)$ is measurable). Then $\exists$ a set $E \subset \mathbb{R}^2$ s.t. (1) $\forall\theta \in [0,\pi), \operatorname{proj}_\theta E \supset A_\theta$ and (2) for a.e. $\theta \in [0,\pi), \mathcal{L}^1((\operatorname{proj}_\theta E) \setminus A_\theta) = 0$.

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**Theorem 5** (A nonlinear variant of Davies's theorem, AC , Alex McDonald , Krystal Taylor )
Let $\Gamma \subset \mathbb{R}^2$ be the graph of a strictly convex $\mathcal{C}^2$ function $[a,b] \to \mathbb{R}$. Then $\forall$ measurable $A \subset \mathbb{R}^2$, $\exists$ a set $K \subset \mathbb{R}^2$ which is a *union of translates of* $\Gamma$ s.t. (1) $K \supset A$ and (2) $\mathcal{L}^2(K \setminus A) = 0$.

# Venetian blinds, digital sundials, and efficient coverings

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**Theorem 5** (A nonlinear variant of Davies's theorem, AC , Alex McDonald , Krystal Taylor )
Let $\Gamma \subset \mathbb{R}^2$ be the graph of a strictly convex $\mathcal{C}^2$ function $[a,b] \to \mathbb{R}$. Then $\forall$ measurable $A \subset \mathbb{R}^2$, $\exists$ a set $K \subset \mathbb{R}^2$ which is a *union of translates of* $\Gamma$ s.t. (1) $K \supset A$ and (2) $\mathcal{L}^2(K \setminus A) = 0$.

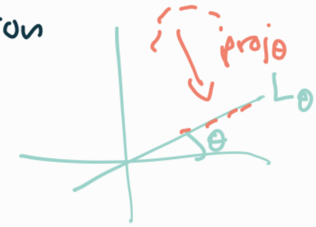
Venetian blind construction:

start:

large projection

end:

small projection

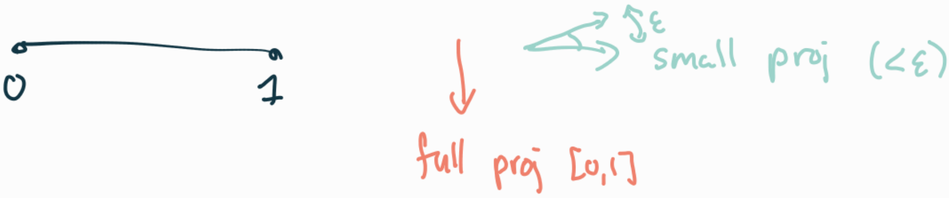



We can use Venetian blinds to prove:

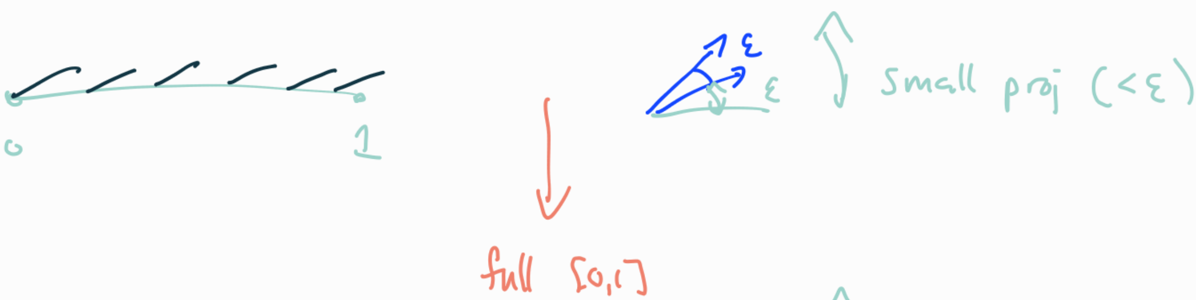
Thm 2: $\exists E \subset \mathbb{R}^2$ s.t. (1) $\text{proj}_0 E \supset [0,1]$

(2) for a.e. $\theta \in [0,\pi)$, $|\text{proj}_\theta E| = 0$

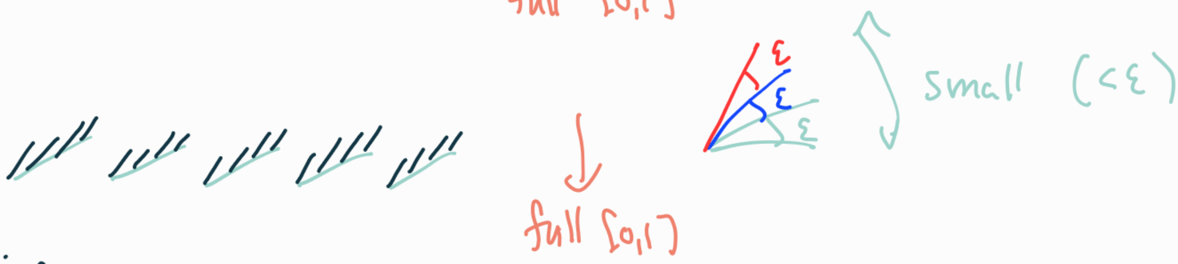
Idea of pf: Start:




Step 2:




step 3:




⋮

Keep iterating. we end up with:

**Lem:** $\forall \varepsilon > 0 \quad \exists\, E \subset \mathbb{R}^2$ s.t. (1) $\text{proj}_0 E \supset [0,1]$

(2) $\forall \theta \notin (-\varepsilon, \varepsilon) \quad |\text{proj}_\theta E| \lesssim \varepsilon$

Repeat this, let $\varepsilon \to 0$ to get

**Thm 2:** $\exists E \subset \mathbb{R}^2$ s.t. (1) $\text{proj}_0 E \supset [0,1]$ (2) $\forall \theta \neq 0$, $|\text{proj}_\theta E| = 0$

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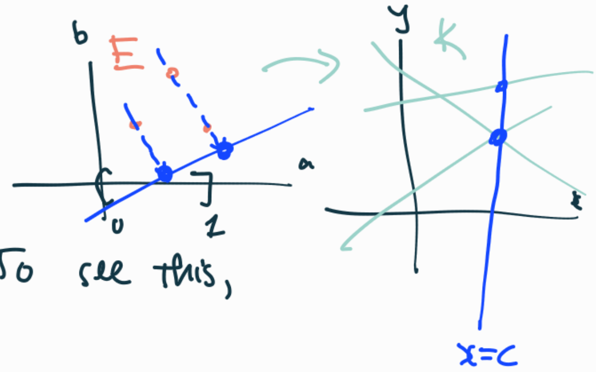
**Thm 1:** $\exists K \subset \mathbb{R}^2$ s.t. (1) $K$ has a line in every direction
(2) $\mathcal{L}^2(K) = 0$

**Pf that Thm 2 ⇒ Thm 1:** Let $E$ be as in Thm 2.

Define $K = \bigcup_{(a,b) \in E} \{(x,y) \in \mathbb{R}^2 : y = ax + b\}$




(1) of Thm 2 $\Rightarrow$ (1) of Thm 1

Also, $\underbrace{\text{(2) of Thm 2}}_{\text{for a.e. } \theta,\ |\text{proj}_\theta E| = 0} \Rightarrow \underbrace{\text{(2) of Thm 1}}_{\mathcal{L}^2(K) = 0}$. To see this,

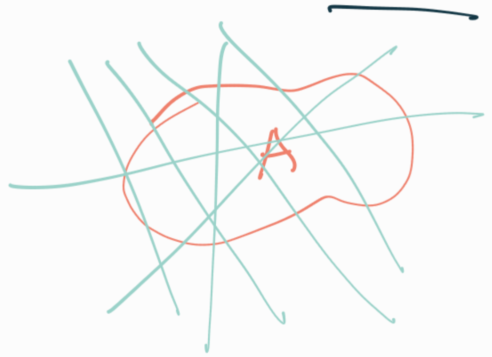
fix $c \in \mathbb{R}$ consider $K \cap \{x = c\}$.

$$K \cap \{x=c\} = \bigcup_{(a,b)\in E} \{(c,y) : y = ac+b\} = \{(c, ac+b) : (a,b) \in E\}$$
$$= \{c\} \times \{ac+b : (a,b) \in E\}$$

This is a proj of E

$ac+b = (a,b)\cdot(c,1)$

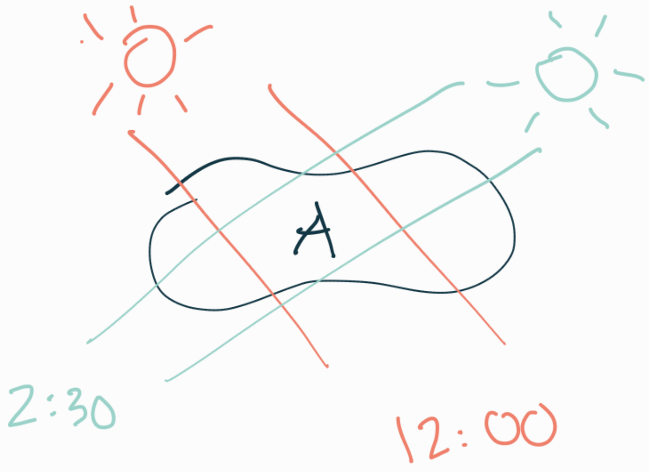
**Thm 3:** Let $A \subset \mathbb{R}^2$. Then $\exists K \subset \mathbb{R}^2$ which is a union of lines
s.t. (1) $K \supset A$ (2) $\mathcal{L}^2(K \setminus A) = 0$.




Note: Thm 1 is actually a special case of thm 3 if you replace $\mathbb{R}^2$ with the real projective plane $\mathbb{RP}^2$.

**Thm 4:** Let $(A_\theta)_{\theta \in [0,\pi)}$ be subsets of $\mathbb{R}$. Then $\exists E \subset \mathbb{R}^2$
s.t. (1) $\forall \theta \quad \text{proj}_\theta E \supset A_\theta$. (2) for a.e. $\theta \quad \mathcal{L}^1((\text{proj}_\theta E) \setminus A_\theta) = 0$

"Digital sundial theorem"




Note: Thm 2 is a special case of Thm 4:

Let $A_0 = [0,1]$ $\quad A_\theta = \emptyset$, $\theta \neq 0$

Thm 4 can be proved using Venetian blinds.

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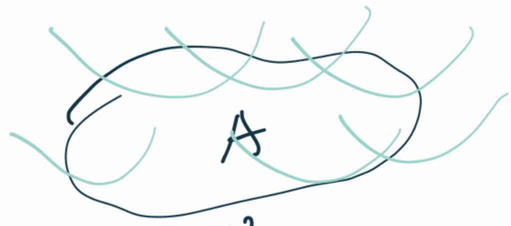
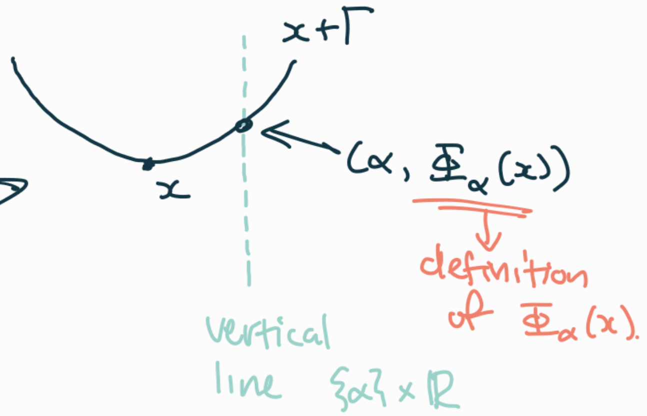
**Theorem 5** (A nonlinear variant of Davies's theorem, AC , Alex McDonald , Krystal Taylor )
Let $\Gamma \subset \mathbb{R}^2$ be the graph of a strictly convex $\mathcal{C}^2$ function $[a,b] \to \mathbb{R}$. Then $\forall$ measurable $A \subset \mathbb{R}^2$, $\exists$ a set $K \subset \mathbb{R}^2$ which is a *union of translates of* $\Gamma$ s.t. (1) $K \supset A$ and (2) $\mathcal{L}^2(K \setminus A) = 0$.





$\Gamma \leftarrow$ Fix $\Gamma$




$\alpha \in \mathbb{R}$. $\quad x \in \mathbb{R}^2$

Define $\Phi_\alpha : \mathbb{R}^2 \to \mathbb{R}$ as follows: $\longrightarrow$

For $E \subset \mathbb{R}^2$, if we define

$$K := \bigcup_{x \in E} (x + \Gamma)$$

$$K \cap (\{\alpha\} \times \mathbb{R}) = \{\alpha\} \times \Phi_\alpha(E)$$

So we want (1) $\forall \alpha \quad \Phi_\alpha(E) \supset A \cap (\{\alpha\} \times \mathbb{R})$

(2) for a.e. $\alpha$, $|\Phi_\alpha(E)| = 0$

the idea of proving this: "$\{\Phi_\alpha\}_\alpha$" behave similarly to "$\{\mathrm{proj}_\theta\}_\theta$" so we can use Venetian blinds

$\mathrm{proj}_\theta^{-1}(\mathrm{proj}_\theta(x))$ $\qquad\qquad$ $\Phi_\alpha^{-1}(\Phi_\alpha(x))$