

46th Summer Symposium
in
Real Analysis

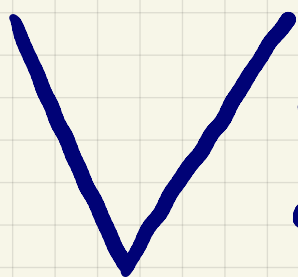
"The Promised Land Symposium"

University of Łódź
Poland

"Hypercyclicity and supercyclicity"

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- Università
- Sepi Studi
- della Compagnia
- "Luigi Vanvitelli"

- Composition dynamical systems
- Keigle's backward shifts
- Hypercyclicity and supercyclicity

•
Defn A composition dynamical system is a quadruple $(X, \mathcal{B}, \mu, \mathcal{T}_g)$, where

1) $(X, \mathcal{B}, \mu)$ is a σ -finite measure space

2) $g: X \rightarrow X$ is an injective bimeasurable transformation

3) $\exists c > 0$ s.t. $\mu(g^{-1}(B)) = c\mu(B)$ for every $B \in \mathcal{B}$

4) $T_g: L^p(X) \rightarrow L^p(X)$, $1 \leq p < \infty$, is the composition operator induced by g , that is

$$T_g: \varphi \mapsto \varphi \circ g$$

N.B. If f is surjective and f^{-1} satisfies 3), then

$$T_f^{-1} = T_{f^{-1}}$$

We deal with "dissipative
continuous dynamical systems
with bounded distortion".

Q: What are they?
What are the extra hypotheses?

- Dissipative?
- Of bounded distortion?

Hopf Decomposition Theorem

Let $(X, \mathcal{B}, \mu)$ be a σ -finite measure space and $f: X \rightarrow X$ be a non-singular transformation. Then, X is the union of two disjoint invariant sets $\mathcal{C}(f)$ and $\mathcal{O}(f)$, called the

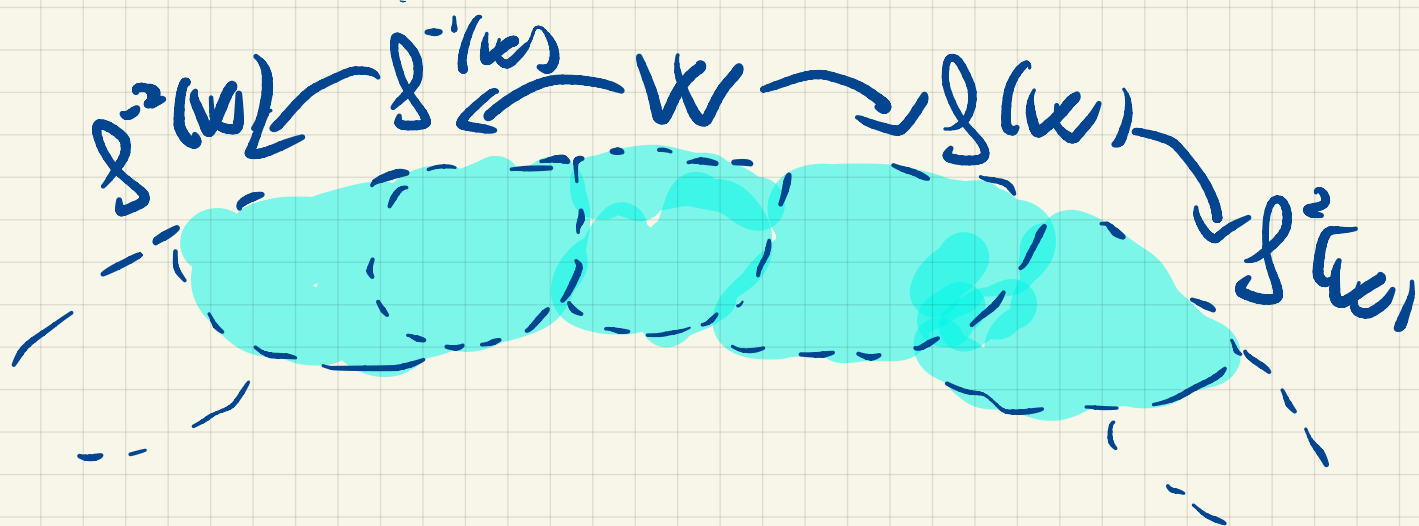
conservative and **dissipative** parts of f , resp., satisfying:

1) $\forall B \subseteq \mathcal{C}(f)$ with $\mu(B) > 0$,
 $\exists n > 0$ s.t. $\mu(B \cap f^{-n}(B)) > 0$

2) $\mathcal{O}(f) = \bigcup_{k \in \mathbb{Z}} f^k(K)$, for

some $K \in \mathcal{B}$.

• Defn of $X = \bigcup_{k \in \mathbb{Z}} g^k(W)$ for some
 $W \in \mathcal{B}$ with $0 < \mu(W) < \infty$



then $(X, \mathcal{B}, \mu, g, T_g)$ is a
 dissipative component dynamical
 system (generated by W).

• Defn If, moreover, $\exists M > 0$ s.t.

$$\forall B \in \mathcal{B}(W) = \{ B \cap W : B \in \mathcal{B} \}$$

$$\frac{1}{M} \frac{\mu(B)}{\mu(W)} \leq \frac{\mu f^k(B)}{\mu f^k(W)} \leq M \frac{\mu(B)}{\mu(W)}$$

then $(X, \mathcal{B}, \mu, f, T_f)$ is

a dissipative composition
dynamical system of
bounded distortion.

Examples?

Plenty!

Take $X = \mathbb{R}$, $\mathcal{B}$ Borel sets,

$$f(x) = x + i, \quad W = [0, 1]$$

$$\text{and } \mu(B) = \int_B h \, d\lambda$$

with h a suitable non-negative

Lebesgue measurable function

$$(e^x, e^{-x}, \frac{1}{2} e^{-|x|}, \frac{1}{\pi(1+x^2)}, \dots)$$

• Defn Let $X = \ell^p(\mathbb{Z})$, $1 \leq p < \infty$.
Let $w = \{w_n\}_{n \in \mathbb{Z}}$ be a bounded
sequence of scalars. Then,
the bilateral weighted shift
 B_w on X is defined by

$$B_w(\{x_n\}_{n \in \mathbb{Z}}) = \{w_{n+1} x_{n+1}\}_{n \in \mathbb{Z}}$$

Q: Is there any relation
between weighted shifts and
invertible composition operators?

A: YES!

Fact every invertible B_k is topologically conjugate via an isometry to a dissipative composition dynamical system of bounded distortion

Hence, every B_k is a dissipative composition dynamical system of bounded distortion.

Q: Given a dissipative composition operator, is it related to some special B_U ?

A: YES!

$$T_g: L^p(X) \rightarrow L^p(X) \\ \varphi \mapsto \varphi \circ g$$

$$B_U: l^p(Z') \rightarrow l^p(Z') \\ \{x_n\} \mapsto \{w_{nn} x_{n_n}\}$$

where

$$w_n = \left(\frac{\mu(g^{n-1}(X))}{\mu(g^n(U))} \right)^{1/p}$$

Let X and Y be separable Banach spaces and let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be operators.

A factor map $\pi: X \rightarrow Y$

is a linear, bounded, surjection for which the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{S} & Y \end{array}$$

" S is a factor of T "

• Thm (D., Dergi, Meinvalle 2022)

B_W is a factor of T_g .

More precisely, if we define $\pi: L^p(X) \rightarrow l^p(Z')$

$$\text{as } (\pi(f))_k = \frac{(\mu g^k(W))^{1/p}}{\mu(W)} \int_W \varphi g^k d\mu,$$

$\forall k \in Z'$, then the following diagram commutes

$$\begin{array}{ccc} L^p(X) & \xrightarrow{T_g} & L^p(X) \\ \pi \downarrow & & \downarrow \pi \\ l^p(Z') & \xrightarrow{B_W} & l^p(Z') \end{array}$$

and π is onto, linear and bounded.

Let X be a separable Banach space and $T: X \rightarrow X$ be a bounded linear operator (briefly, an operator).

• Defn The operator T is hypercyclic if it admits a hypercyclic vector, i.e. if there exists $x \in X$ such that $\{T^n(x) : n \geq 0\}$ is dense in X .

• Defn The operator T is topologically transitive if for any pair of non-empty open subsets U, V of X there is $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$.

• RHK By the Birkhoff transitivity theorem, in the separable context, transitivity and hypercyclicity are the same thing.

• Defn An operator T is cyclic if it admits a cyclic vector, i.e. if there is $x \in X$ such that $\text{span}\{T^n(x) : n \geq 0\}$ is dense in X .

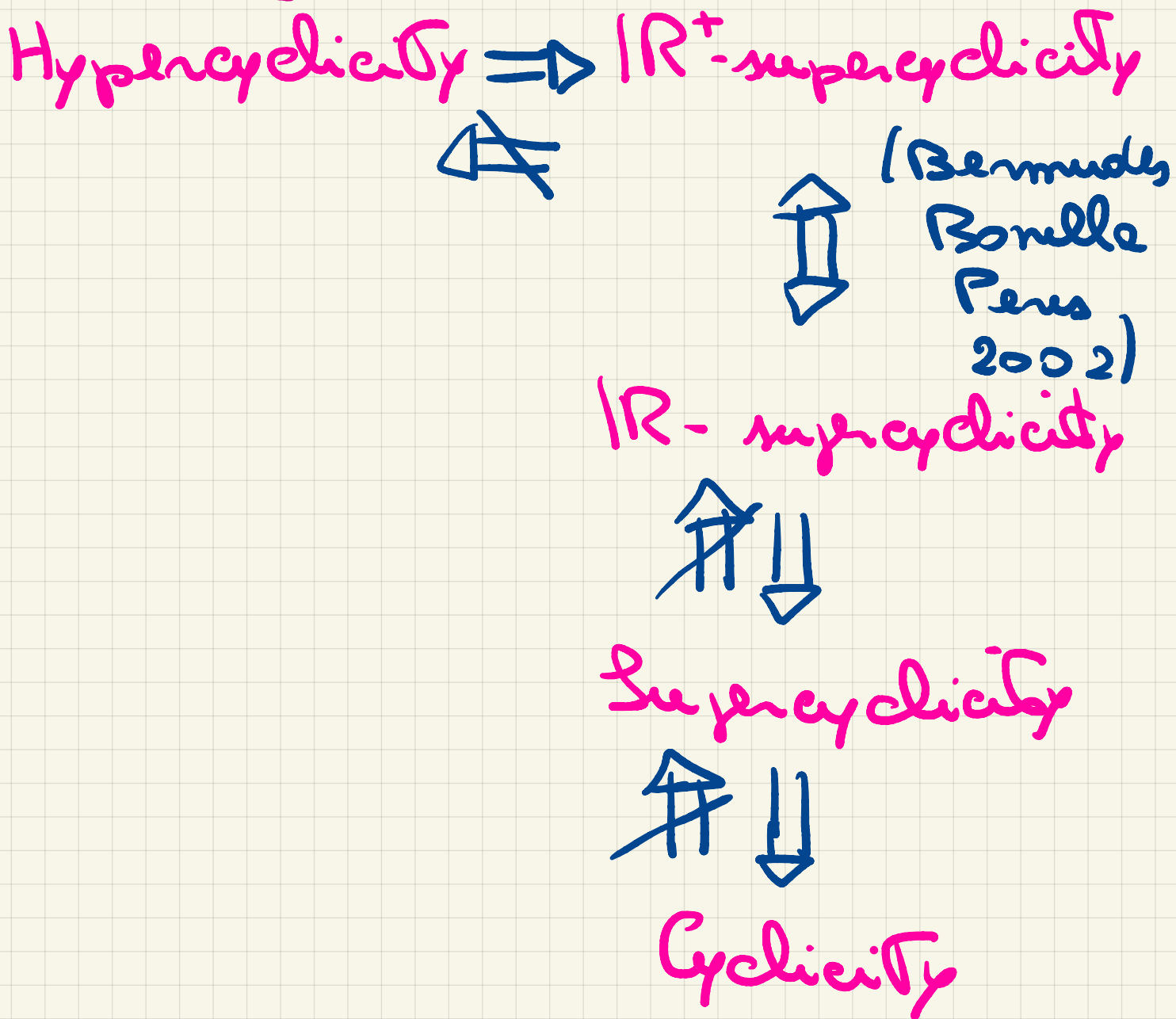
• Defn An operator T is $\mathbb{C}$ -cyclic (or, simply, cyclic) if it admits a $\mathbb{C}$ -cyclic vector (or, simply, cyclic vector), i.e. if there exists $x \in X$ such that $\{ \lambda T^n(x); n \geq 0, \lambda \in \mathbb{C} \}$ is dense in X .

• Defn An operator T is $\mathbb{R}$ -cyclic if it admits a $\mathbb{R}$ -cyclic vector, i.e. if there is $x \in X$ such that $\{ \lambda T^n(x); n \geq 0, \lambda \in \mathbb{R} \}$ is dense in X .

By replacing $\mathbb{R}$ with $\mathbb{R}^+$ we obtain the definition of $\mathbb{R}^+$ -cyclic

Question: What are the relations between these notions?

Answer:



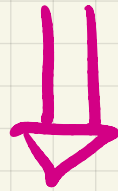
Thm (analogue of Birkhoff transitivity theorem) (D, Hamiele 2024)

The following are equivalent:

- (i) T is mixing (resp. $\mathbb{R}$ mixing, $\mathbb{R}^+$ mixing)
- (ii) $\forall U, V$ non-empty open subsets of X , $\exists n \geq 0$ and $\lambda \in \mathbb{P}$ (resp. $\lambda \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$) s.t.
 $\lambda T^n(U) \cap V \neq \emptyset$.

RMN All the previous
properties are carried over
by factors.

T hypercyclic (resp. surcyclic)



S hypercyclic (resp. surcyclic)

Question: How about the other way around?

Answer: In some cases, "yes".

Let $(X, \mathcal{B}, \mu, f, T_f)$ be
a dissipative compact
dynamical system of bounded
distortion:

$$\bullet T_f: L^p(\mu) \rightarrow L^p(\mu)$$

$$\varphi \mapsto \varphi \circ f$$

$$\bullet X = \bigcup_{n \in \mathbb{Z}'} f^n(W)$$

$$(W \in \mathcal{B} \text{ and } 0 < \mu(W) < \infty)$$

Then the backward shift

$$B_W: l^p(\mathbb{Z}') \rightarrow l^p(\mathbb{Z}')$$

$$\underline{x} = (x_n) \mapsto (w_{n+1} x_{n+1})_{n \in \mathbb{Z}'}$$

$$\text{with } w_n = \left(\frac{\mu(f^{n-1}(W))}{\mu(f^n(W))} \right)^{\frac{1}{n}}$$

is a factor of T_f and

a factor map is Γ given by

$$\Gamma(\varphi)(n) = \frac{(\mu(g^n(W)))^{1/p}}{\mu(W)} \int_W \varphi \circ g^n d\mu,$$

$$\forall \varphi \in L^p(\mu).$$

$$\begin{array}{ccc} X & \xrightarrow{T_g} & X \\ \Gamma \downarrow & & \downarrow \Gamma \\ \ell^p(Z) & \xrightarrow{B_\mu} & \ell^p(Z') \end{array}$$

Hence:

T_g hypercyclic (resp. mixing)



B_μ hypercyclic (resp. mixing) 22

For hypercyclicity,
the converse also holds!

Thm (D, Darzi, Meinert, 2022)
 B_∞ hypercyclic $\Rightarrow T_g$ hypercyclic.

Pf Main ingredient:

General characterisation of hypercyclicity for T_g (Bayart, Darzi, Peris 2018)

Let $(X, \mathcal{B}, \mu, g, T_g)$ be a composition dynamical system.

The T_g is hypercyclic iff, $\forall \varepsilon > 0$
 $\forall B \in \mathcal{B}$ of finite measure,

$\exists B' \subseteq B$ and $k \geq 1$ s.t.

$\mu(B, B') < \varepsilon$, $\mu(g^k(B')) < \varepsilon$
and $\mu(g^{-k}(B')) < \varepsilon$.

Q: How about π -cyclicity
and $1R$ - π -cyclicity?

• Thm (D. Maue 2024)

The following are equivalent:

- (I) T_f is mixing
- (II) B_μ is mixing
- (III) B_μ is $\mathbb{R}$ -mixing
- (IV) T_f is $\mathbb{R}$ -mixing

Pf Main ingredient: General characterization of $\mathbb{R}$ -mixing
 T_f (D. Maue 2024)

The composition operator T_f is $\mathbb{R}$ -mixing iff $\forall \varepsilon > 0, \forall B \in \mathcal{B}$
of finite measure, $\exists B' \subseteq B, k \geq 1$
and $\lambda > 0$ s.t. $\mu(B, B') < \varepsilon$
 $\mu(f^k(B')) < \lambda \varepsilon$ and $\mu(f^{-k}(B')) < \lambda \varepsilon$

Thm (Bayer-Matthewson)

Let B_w be bilateral weighted backward shift on $\ell^p(\mathbb{Z})$ ($1 \leq p < \infty$) with weight sequence $w = (w_n)_{n \in \mathbb{Z}}$.

Then B_w is aperiodic

iff, for any $q \geq 0$,

$$\lim_{n \rightarrow +\infty} \left\{ \frac{1}{w_1 \cdots w_{n+q}} \cdot (w_0 \cdots w_{-n+q+1}) \right\} \neq 0$$

Cor (D., Meinke, 2024)

T_g is supercyclic iff, $\forall q \geq 0$,

$$\lim_{n \rightarrow \infty} \mu(g^{q-n}(U)) \cdot \mu(g^{q+n}(U)) = 0.$$

□

Exmp Let $X = \mathbb{R}$, $\mathcal{B} = \mathcal{B}(\mathbb{R})$,

$$g: \mathbb{R} \rightarrow \mathbb{R}, \quad U = [0, 1[\text{ and } \\ x \mapsto x+1$$

define

$$\mu(B) = \int_B h(x) dx, \quad \forall B \in \mathcal{B},$$

$$\text{where } h(x) = \frac{1}{2} e^{-|x|}$$

Then,

$$\mu(W) = \frac{1}{2} \left(1 - \frac{1}{2}\right)$$

and, for $n \geq 1$,

$$\begin{aligned}\mu(\mathcal{I}^n(W)) &= \int_n^{n+1} h(x) dx = \frac{1}{2} \int_n^{n+1} e^{-x} dx \\ &= \frac{1}{2} e^{-n} \left(1 - \frac{1}{e}\right)\end{aligned}$$

and

$$\begin{aligned}\mu(\mathcal{I}^{-n}(W)) &= \frac{1}{2} \int_{-n}^{-n+1} e^{-x} dx \\ &= \frac{1}{2} e^{-n+1} \left(1 - \frac{1}{e}\right).\end{aligned}$$

Then, for each $q \in U$, $n \geq 1$,

$$\begin{aligned}\mu(J^{q+n}(U)) &\leq c^q \mu(J^n(U)) \\ &= c^q \frac{e^{-n}}{2} \left(1 - \frac{1}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\mu(J^{q-n}(U)) &\leq c^q \mu(J^{-n}(U)) \\ &= c^q \frac{e^{-n+1}}{2} \left(1 - \frac{1}{2}\right).\end{aligned}$$

Therefore,

$$0 \leq \mu(J^{q+n}(U)) \cdot \mu(J^{q-n}(U))$$

$$\leq c^{2q} \frac{e^{-2n+1}}{4} \left(1 - \frac{1}{2}\right)^2 \longrightarrow 0$$

as $n \rightarrow +\infty$,

Therefore T_g is $\mathbb{C}$ -mixing.

Open Is it always true

T_g Γ -mixing $\Leftrightarrow P_u$ Γ -mixing?

Defn Given $\Gamma \subseteq \mathbb{C}$, $T: X \rightarrow X$
is Γ -mixing if T has
a Γ -mixing vector that is
 $\exists x \in X$ such that

$$\overline{\{ \lambda T^n(x) : \lambda \in \Gamma, n \geq 0 \}} = X$$

(Covenier, Ernst, Meneš, 2016)