Perfect cliques with respect to infinitely many relations

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Cliques and independent sets

Let R be a relation on a set X. Let n be the arity of R.
A set $C \subseteq X$ is called an R-clique if $R(x_1, \dots, x_n)$ holds whenever $x_1, \dots, x_n \in C$ are pairwise distinct.
A set $I \subseteq X$ is called R-independent if $\neg R(x_1, \dots, x_n)$ holds whenever $x_1, \dots, x_n \in C$ are pairwise distinct.

Let $\mathcal{R}$ be a family of relations on a set X.
A set $C \subseteq X$ is called an $\mathcal{R}$-clique if it is an R-clique for every $R \in \mathcal{R}$.
A set $I \subseteq X$ is called $\mathcal{R}$-independent if it is R-independent for every $R \in \mathcal{R}$.

Motivation

Theorem (Feng 1993)

Let X be an analytic subset of a Polish space. Let $R \subseteq X^2$ be a symmetric open set which does not intersect the diagonal. Then either

- $X = \cup_{n \in \omega} X_n$, *where* X_n *is* R*-independent for every* $n \in \omega$,

or else

- *there exists a perfect set* P *which is an* R*-clique.*

We want to find sufficient conditions for the existence of perfect cliques, but...

1. relations which are not open
2. non-binary relations; more than one relation
3. more general spaces

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Remark (Blass)

The theorem above fails when 2 *is replaced by* 3.

By taking $R = X^2 \setminus \{(x, x) : x \in X\}$ we obtain...

Perfect Set Theorem (Souslin)

Let X be an analytic subset of a Polish space. Then either X is countable, or else X contains a perfect set.

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Theorem (Mycielski 1964)

Let X be a Polish space without isolated points and let $\mathcal{R}$ be a countable family of co-meager relations on X. Then there exists a perfect $\mathcal{R}$-clique.

Theorem (Shelah 1999)

The following statement is not decided by ZFC + $(2^{\aleph_0} > \aleph_{\omega_1})$:

Let R be an analytic relation on a Polish space. Suppose that there exists an R-clique of cardinality $> \aleph_1$. Then there exists a perfect R-clique.

Theorem (Shelah 1999; Kubiś & Vejnar 2012)

There exists a σ-compact symmetric binary relation R on the Cantor space such that

1. *there exists an R-clique of cardinality $\aleph_1$,*
2. *there are no R-cliques of cardinality $> \aleph_1$,*
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Main result

Theorem (Kubiś & D. 2016)

Let X be a completely metrizable space of weight $\kappa \geq \aleph_0$ and let $\mathcal{R}$ be a countable family of G_δ relations on X. Then either

- *there exists an ordinal $\gamma < \kappa^+$ such that the Cantor-Bendixson rank of every $\mathcal{R}$-clique is $\leq \gamma$,*

or else

- *there exists a perfect $\mathcal{R}$-clique.*

Remark

The theorem fails if we replace the family $\mathcal{R}$ by a single binary F_σ relation [Shelah 1999; Kubiś & Vejnar 2012].

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Corollary

Let X be a completely metrizable space and let $\mathcal{R}$ be a countable family of G_δ relations on X. Suppose that there exists a nonempty $\mathcal{R}$-clique without isolated points. Then there exists a perfect $\mathcal{R}$-clique.

Corollary

Let X be an analytic subset of a Polish space and let $\mathcal{R}$ be a countable family of G_δ relations on X. Suppose that there exists an uncountable $\mathcal{R}$-clique. Then there exists a perfect $\mathcal{R}$-clique.

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Free subgroups of Polish groups

Theorem (Głąb & Strobin 2015)

Let $G = \prod_{n\in\omega} G_n$, where each G_n is a countable group. If G contains an uncountable free subgroup then it also contains a free subgroup of cardinality $2^{\aleph_0}$.

Theorem

Let G be a Polish group. Then either all free subgroups of G are countable, or else G contains a perfect set generating a free subgroup.

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Proof:

For each nonempty word $w = w(g_1, \dots, g_n)$ on G, we put

$$R_w = \{(g_1, \dots, g_n) \in G^n \colon w(g_1, \dots, g_n) \neq 1\}.$$

Then each R_w is an open relation on G. Further, a subset of G generates a free subgroup iff it is an R_w-clique for every w. Now apply (a corollary of) our main result. □

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Similarly, one can prove...

Theorem

Let G be a completely metrizable topological group containing a nonempty set, without isolated points, generating a free subgroup. Then G contains a perfect set generating a free subgroup.

... and other variants, e. g. for free abelian subgroups, torsion-free subgroups, etc.

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