Remarks on center of distances

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46TH SUMMER SYMPOSIUM IN REAL ANALYSIS
The Promised Land Symposium
June 17–21, 2024

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$$S(X) := \{\alpha : \forall_{x \in X} \exists_{y \in X}\ \rho(x, y) = \alpha\}$$

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If $A \subset \mathbb{R}$ and ρ is the Euclidean metric then

$$S(A) := \{\alpha : \forall_{x \in A} (x - \alpha \in A \text{ or } x + \alpha \in A)\}$$

W. Bielas, S. Plewik, M. Walczyńska, *On the center of distances*, Eur. J. Math. (2018), 4, 687-698

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- in particular, for any compact set A, $S(A) = S(A - \min A)$;
- if $A \subset [0, \infty)$ and $0 \in A$ then $S(A) \subset A$

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$$S\left([0, a] \cup [3a, 4a] \cup [10a, 11a]\right) = \left[0, \frac{a}{2}\right]$$

Important example

Theorem 1

Let $(a_n) \searrow 0$ *be a summable sequence and*

$$E(a_n) := \left\{ x \in \mathbb{R} : \exists_{M \subset \mathbb{N}} \ \ x = \sum_{n \in M} a_n \right\}$$

be a set of subsums of the sequence (a_n)*. Then*

$$\{a_n : n \in N\} \subset S(E(a_n)).$$

In particular

$$S\left(C_{1/3}\right) = S\left(E\left(\frac{2}{3^n}\right)\right) = \{0\} \cup \left\{ \frac{2}{3^n} : n = 1, 2, 3, \dots \right\}$$

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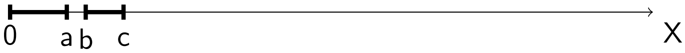

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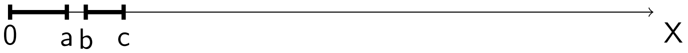

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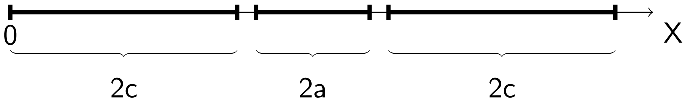

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But $y \in \bigcup_{t\in T} A_t$ so $\alpha \in S\left(\bigcup_{t\in T} A_t\right)$.

Proposition 4

Let $T := \mathbb{N}$ *or* $T := \{1, ..., n\}$ *and* $\{A_t : t \in T\}$ *be a sequence of nonempty subsets of* $[0, Z]$*. Then there exists a set* $A \subset \mathbb{R}$ *such that* $S(A) = \bigcap_{t \in T} S(A_t)$*.*

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Using **Lemma 3** we have

$$\bigcap_{t \in T} S(A_t) = \bigcap_{t \in T} S\left(\widetilde{A_t}\right) \subset S\left(\bigcup_{t \in T} \widetilde{A_t}\right) = S(A).$$

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Moreover, the sets A_t are so spread out that

$$S(A) \subset \bigcap_{t \in T} S(A_t).$$

□

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From **Lemma 2** we know that for any $t \in T$ there exists a set $A_t \subset [0,6]$ such that $S\left(A_t\right) = [0,1] \setminus G_t$ so

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From **Proposition 4** there exists a set A such that $S(A) = B$.

A. Bartoszewicz, M. Filipczak, G. Horbaczewska, S. Lindner, F. Prus-Wiśniowski, *On the operator of center of distances between the spaces of closed subsets of the real line* Topol. Methods Nonlinear Anal. Advance Publication 1–15, 2023.

Question 1 - boundness

Sets constructed using our method take up a lot of space on the line, they can even be unbounded.

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Is it true that for any compact set $B \subset [0, 1]$ containing 0 there exists a compact set A such that $S(A) = B$?

The set $S(A)$ can be nonmeasurable and can be a set without the Baire property.

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It seems to be interesting when, i.e. under which assumption on A, we get Borel or measurable $S(A)$.

- If A is closed then $S(A)$ is closed.

Partial answers

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- If A is closed then $S(A)$ is closed.
- If A is open then $S(A)$ is a G_δ set.
- There exists an open set A such that $S(A)$ is not a F_σ set.

Question 3 - existence

Does there exist a set B which is not a center of distances for any $A \subset \mathbb{R}$?