The Borel complexity of sets of ideal limit points

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The talk is based on a join work with Adam Kwela and Paolo Leonetti

Redefinitions

$\omega = \mathbb{N}$ is the set of all natural numbers

X will stand for an uncountable Polish space (i.e. separable completely metrizable topological space)

Ideals on ω

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ideal on ω if

1. $\emptyset \in \mathcal{I}$ and $\omega \notin \mathcal{I}$,
2. $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
3. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
4. $\mathcal{I}$ contains all finite subsets of ω.

Example

1. $\mathrm{Fin} = \{A \subseteq \omega : A \text{ is finite}\}$
2. $\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty\right\}$ — the summable ideal
3. $\mathcal{I}_d = \left\{A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n} = 0\right\}$ — the density zero ideal

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PART 1: Finding one convergent subsequence

Convergent subsequences

Theorem (Bolzano-Weierstrass)

For every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is an $A \notin \mathrm{Fin}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

Theorem (Folklore)

For every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is an $A \notin \mathcal{I}_{1/n}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

Theorem (Fridy, 1993)

There exists a sequence $(x_n)_{n\in\omega}$ in $[0,1]$ such that for every $A \notin \mathcal{I}_d$ the subsequence $(x_n)_{n\in A}$ is not convergent.

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Finite Bolzano-Weierstrass property

Definition (F.-Mrożek-Recław-Szuca, 2007)

An ideal $\mathcal{I}$ has finite Bolzano-Weierstrass property (FinBW property) if for every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

- Fin and $\mathcal{I}_{1/n}$ have the FinBW property
- $\mathcal{I}_d$ does not have the FinBW property

Definition

An ideal $\mathcal{I}$ is F_σ if the set $\{\mathbf{1}_A : A \in \mathcal{I}\}$ is an F_σ subset of the Cantor space $2^\omega = \{0,1\}^\omega$.

The same for $F_{\sigma\delta}$, Borel, analytic, and other topological properties.

Theorem (F.-Mrożek-Recław-Szuca, 2007)

Every F_σ ideal has FinBW property.

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If X is not compact, then there is a sequence $(x_n)_{n\in\omega}$ in X such that for every $A \notin \mathcal{I}$ the subsequence $(x_n)_{n\in A}$ is not convergent in X.

Katětov order (Katětov, 1968)

$\mathcal{I} \leq_K \mathcal{J} \iff$ there exists $f:\omega\to\omega$ such that

$$\forall A \subseteq \omega\, (A \in \mathcal{I} \implies f^{-1}[A] \in \mathcal{J}).$$

The ideal *conv*

$conv = \{A \subseteq \mathbb{Q} : A \text{ has at most finitely many limit points in } \mathbb{R}\}$

Theorem (Meza-Alcántara, 2009)

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PART 2: Finding all convergent subsequence

Sets of ideal limit points of a sequence

Set of ideal limit points of a sequence

$$\Lambda_{\mathcal{I}}((x_n)_{n\in\omega}) = \{p \in X : \exists\, A \notin \mathcal{I}\ ((x_n)_{n\in A} \to p)\}$$

Theorem (Folklore)

$\Lambda_{\mathrm{Fin}}(x_n)$ is a closed set for every sequence (x_n).

Theorem (Balcerzak-Leonetti, 2019)

If an ideal $\mathcal{I}$ is F_σ, then $\Lambda_{\mathcal{I}}(x_n)$ is closed for every sequence $(x_n)_{n\in\omega}$.

Theorem (Kostyrko-Mačaj-Šalát-Strauch, 2001)

- For every nonempty F_σ set $F \subseteq [0,1]$ there exists a sequence $(x_n)_{n\in\omega}$ in $[0,1]$ such that $F = \Lambda_{\mathcal{I}_d}(x_n)$.
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PART 3: Borel complexity of ideal limit sets $\Lambda_{\mathcal{I}}(x)$

Family of all sets of ideal limit points of sequences

Recall: set of ideal limit points of a sequence

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Family of all sets of ideal limit points of sequences

For a space X we write:

$$\Lambda_{\mathcal{I}}(X) = \{\Lambda((x_n)_{n\in\omega}) : \text{ for each sequence } (x_n) \text{ in } X\}$$

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Some examples

Theorem (Folklore)

$\Lambda_{\mathrm{Fin}}(X) = \Pi^0_1(X)$ (all closed subsets of X)

Theorem (Folklore)

If $\mathcal{I}$ is a maximal ideal, then

1. $\Lambda_{\mathcal{I}}(X) = \{\{x\} : x \in X\} \cup \{\emptyset\}$
2. $\Lambda_{\{\emptyset\}\otimes\mathcal{I}}(X) = \{B : B \text{ is countable}\}$
3. $\Lambda_{\mathrm{Fin}\oplus(\{\emptyset\}\otimes\mathcal{I})}(X) = \{A \cup B : A \text{ is closed and } B \text{ is countable}\}$.

Theorem

1. $\Lambda_{\mathcal{I}_{1/n}}(X) = \Pi^0_1(X)$ (Balcerzak-Leonetti, 2019)
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Tools for characterizations

Definition

A family $\{A_s : s \in 2^{<\omega}\}$ of subsets of ω is called an $\mathcal{I}$-scheme if for every $s \in 2^{<\omega}$

1. $A_s \notin \mathcal{I}$,
2. $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$,
3. $A_{s^\frown 0} \cup A_{s^\frown 1} \subseteq A_s$.

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$B_{\mathcal{I}}(\mathcal{A}) = \{x \in 2^\omega : \neg(\exists C \notin \mathcal{I}\ \forall n \in \omega\ |C \setminus A_{x \restriction n}| < \omega)\}$

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- $\mathcal{I} \in P(\Pi^0_1)$ if there is an $\mathcal{I}$-scheme $\mathcal{A}$ with $B_{\mathcal{I}}(\mathcal{A}) = \emptyset$
- $\mathcal{I} \in P(\Sigma^0_2)$ if there is an $\mathcal{I}$-scheme $\mathcal{A}$ with $B_{\mathcal{I}}(\mathcal{A}) = \{(0, 0, \dots)\}$
- $\mathcal{I} \in P(\Pi^0_3)$ if there is an $\mathcal{I}$-scheme $\mathcal{A}$ with $B_{\mathcal{I}}(\mathcal{A}) = \mathbb{Q}(2^\omega)$.

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Open and closed sets

Theorem

(1) The following conditions are equivalent.

1. $\mathcal{I} \in P(\Pi^0_1)$.
2. $\Pi^0_1(X) \subseteq \Lambda_{\mathcal{I}}(X)$.
3. $\Lambda_{\mathcal{I}}(X)$ contains an analytic set which is not countable.

(2)

- $\mathcal{I} \in P(\Pi^0_1) \setminus P(\Sigma^0_2)$ for every F_σ ideal $\mathcal{I}$
- $\mathcal{I} \in P(\Pi^0_1)$ for every ideal $\mathcal{I}$ with the Baire property.
- There exists an ideal $\mathcal{I} \in P(\Pi^0_1)$ which does not have the Baire property (at least under CH).

Remark

The inclusion $\Pi^0_1(X) \subseteq \Lambda_{\mathcal{I}}(X)$ was earlier proved for

- F_σ ideals by Balcerzak-Leonetti (2019)
- ideals with Baire property by He-Zang-Zang (2022)

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The inclusion $\Pi^0_1(X) \subseteq \Lambda_{\mathcal{I}}(X)$ was earlier proved for

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Open and closed sets

Theorem

(1) The following conditions are equivalent.

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Theorem (Folklore)

$\Lambda_{\mathcal{I}}(X) \neq \Sigma^0_1(X)$ for any ideal $\mathcal{I}$.

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(2) $\mathcal{I} \in P(\Sigma^0_2)$ for ideals with the hereditary Baire prop. which aren't P^+.

In particular,

- $\{\emptyset\} \otimes \mathrm{Fin} \in P(\Sigma^0_2)$
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PART 4: Borel complexity of ideal vs. Borel complexity of $\Lambda_{\mathcal{I}}(x)$

Borel complexity of ideals

Theorem

(1) For each $\alpha \geq 3$ there is an ideal $\mathcal{I} \in \Sigma^0_\alpha \setminus \Pi^0_\alpha$ such that

$$\Lambda_{\mathcal{I}}(X) = \Pi^0_1(X).$$

(2) If $\mathcal{I}$ is Σ^0_2, then $\Lambda_{\mathcal{I}}(X) = \Pi^0_1(X)$.

(3) If $\mathcal{I}$ is Π^0_3, then one of the following items holds.

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Borel complexity of ideals

Definition

An ideal $\mathcal{I}$ is called a Farah ideal if there is a family of compact hereditary sets $\{C_n : n < \omega\}$ such that $\mathcal{I} = \{A \subseteq \omega : \forall n < \omega \, \exists m < \omega \, (A \setminus [0, m) \in C_n)\}$.

It is known that every Farah ideal is Π^0_3.

Theorem (He-Zang-Zang, 2022)

If $\mathcal{I}$ is a Farah ideal, then $\Lambda_{\mathcal{I}}(X) \subseteq \Sigma^0_2(X)$

Corollary

If $\mathcal{I}$ is a Farah ideal, then one of the following items holds.

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Question (Farah, 2004)

Is every Π^0_3 ideal a Farah ideal?

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$F_{\sigma\delta}$ and $G_{\delta\sigma}$ sets

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But wait, we haven't defined the property $P(\Sigma^0_4)$!