A universal operator on ℓ_1

Joanna Garbulińska-Węgrzyn

Faculty of Mathematics
Jan Kochanowski University in Kielce

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1. A separable Banach space X is called Gurariĭ if for any finite-dimensional Banach spaces $A \subseteq B$ and any $\varepsilon > 0$, any isometric embedding $i : A \to X$ extends to an ε-isometric embedding $\bar{i} : B \to X$.
2. An operator $U : V \to W$ between Banach spaces is defined to be universal if for every operator $T : X \to Y$ with $\|T\| \leq \|U\|$, there exist linear isometric embeddings $i : X \to V$, $j : Y \to W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\;U\;} & W \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ X & \xrightarrow{\;T\;} & Y \end{array}$$

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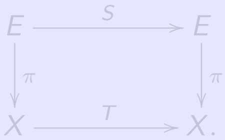
1. Let T and S be linear operators acting on Banach spaces X and E, where T and S are self-maps of X and E, respectively. Then T is a linear lifting of S, and S is a linear factor of T if there is a map π from E onto X such that $T\pi = \pi S$, i.e. the following diagram is commutative:



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$$\begin{array}{ccc} E & \xrightarrow{\;S\;} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\;T\;} & X. \end{array}$$

1. Let us start with the following recent result, due to Darji and Matheron [1]:

   (K) There exists a non-expansive linear operator $U : \ell_1 \to \ell_1$ such that for every separable Banach space E, for every non-expansive operator $f : E \to E$ there is a non-expansive surjective linear operator $q : \ell_1 \to E$ satisfying $f \circ q = q \circ U$.

$$\begin{array}{ccc} \ell_1 & \xrightarrow{\ U\ } & \ell_1 \\ \downarrow q & & \downarrow q \\ E & \xrightarrow{\ f\ } & E. \end{array}$$

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$$\begin{CD} \ell_1 @>U>> \ell_1 \\ @VqVV @VVqV \\ E @>f>> E. \end{CD}$$

1. f lifts to U if there is q as above, namely, satisfying $f \circ q = q \circ U$.
2. $\ell_1 S$ is the space of all functions x with domain S and the range in the scalar field, either real or complex, so that

$$\|x\| := \sum_{s \in S} |x(s)| < +\infty.$$

3. Given $a \in S$, we denote by $\widehat{a}$ the Kronecker delta of a, namely, $\widehat{a}(s) = 1$ if $s = a$ and $\widehat{a}(s) = 0$ otherwise.
4. The collection $\{\widehat{s}\}_{s \in S}$ is the standard basis of $\ell_1 S$. This is a monotone Schauder basis satisfying $\|\widehat{s} - \widehat{t}\| = 2$ for every $s \neq t$ in S.

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1. By an operator we mean a bounded linear map.
2. An operator $f : \ell_1 A \to \ell_1 B$ is basic if there is a map $g : A \to B$ such that $f\widehat{a} = \widehat{ga}$ for every $a \in A$.
3. A basic embeddings is basic operators as above, induced by an injective map $g : A \to B$. Given a basic embedding $e : \ell_1 A \to \ell_1 B$, we denote by $\widehat{e}$ the unique map from A to B such that $e(\widehat{a}) = \widehat{\widehat{e}(a)}$ for every $a \in A$.
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1. An operator $A : \ell_1(S) \to \ell_1(S)$ is called 0-basic if it is defined by a self-map $\phi : S \to S$ in the sense that $A\hat{s} = \widehat{\phi(s)}$ or $A\hat{s} = 0$ for every $s \in S$.
2. Let us denote a set of 0-basic operators on ℓ_1 as $\mathcal{BO}_0(\ell_1)$, obviously $\mathcal{BO}(\ell_1) \subset \mathcal{BO}_0(\ell_1)$.
3. Let $(T_n)_{n\in\omega}$ be a sequence of linear operators on the Banach space X which converges to some operator T on for all $x \in X$ in the weak operator topology if for all continuous linear functional F on X we have $F(T_n x) \to F(Tx)$.

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1. Weak topology is the coarsest topology which makes all the linear functionals in the dual space of X to be continuous.
2. Let us define the weaker natural topology (called the linear Hausdorff topology) on the space $\mathcal{B}(\ell_1)$ of bounded linear operators on ℓ_1 as

$$\tau_{\mathcal{BO}_0(\ell_1)} = \ell_1^*,$$

which is generated by the neighborhood of zero in $\mathcal{B}(\ell_1)$:

$$\mathcal{U}(E, \varepsilon) = \{T \in \mathcal{B}(\ell_1) : \forall n, m \in E \; |e_m^*(T(e_n))| < \varepsilon\},$$

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Remark

Observe that operator T is not 0-basic ($T \notin \mathcal{BO}_0(\ell_1)$) if there exists a natural number i such that

$$\ell_1 \ni T(e_i) \notin \{e_j : j \in \mathbb{N}\} \cup \{0\}.$$

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The set $\mathcal{BO}_0(\ell_1)$ is closed in the topology $\tau_{\mathcal{BO}_0(\ell_1)}$.

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Corollary

The set $\mathcal{BO}_0(\ell_1)$ is nowhere dense in any linear topology on $\mathcal{B}(\ell_1)$.

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1. In [3] Kubiś proved that every non-expansive linear operator $f : E \to E$ on Banach space E lifts to a basic operator on ℓ_1.
2. The existence of a surjectively universal operator U can be reduced to proving the existence of a surjectively universal self-mapping of a countable set.

Theorem

Let S, T be nonempty sets, $f : S \to S$, $g : T \to T$ and $p : T \to S$ be such that p is a surjection and $p \circ g = f \circ p$. Then the diagram

$$\begin{array}{ccc} \ell_1(T) & \xrightarrow{\ell_1(g)} & \ell_1(T) \\ \downarrow{\scriptstyle \ell_1(p)} & & \downarrow{\scriptstyle \ell_1(p)} \\ \ell_1(S) & \xrightarrow{\ell_1(f)} & \ell_1(S). \end{array}$$

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1. In other words, under the assumptions above, $\ell_1(f)$ lifts to $\ell_1(g)$.
2. Given a self-map $f : S \to S$, an f-orbit is any set of the form $\{f^n(x) : n \in \omega\}$, where $x \in S$.
3. Let $\mu : \omega^2 \to \omega^2$ be defined by $\mu(m, n) = (m, n + 1)$, $(m, n) \in \omega^2$.
4. Clearly, μ is one-to-one and all of its orbits are infinite.

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μ is surjectively universal. Namely, given a nonempty countable set S, given a mapping $f : S \to S$, there is a surjection $q : \omega^2 \to S$ such that $q \circ \mu = f \circ q$.

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Theorem

The operator $\ell_1(\mu)$ is surjectively universal in the category of separable Banach spaces.

Recall that ℓ_1 is actually a functor from the category of sets into the category of Banach spaces with non-expansive operators, that is left adjoint to the forgetful functor assigning to a Banach space E its unit ball B_E - $\ell_1 S$ is the free Banach space over the set S, in the same sense as the free group and similar algebraic objects.

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Let us now define our main category $\mathcal{S}$ leading to a generic operator on ℓ_1.
The objects are non-expansive operators of the form $f : \ell_1 A \to \ell_1 B$, where A, B are finite sets. The arrows are pairs of basic embeddings $\langle e_0, e_1 \rangle$ commuting with the operators in the usual way, as in the diagram below.

$$\begin{array}{ccc} \ell_1 A & \xhookrightarrow{\;e_0\;} & \ell_1 A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \ell_1 B & \xhookrightarrow{\;e_1\;} & \ell_1 B'. \end{array}$$

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We also define the (relevant for us) subcategory $\mathcal{B}$, as follows: the objects of $\mathcal{B}$ are self-operators $f : \ell_1 A \to \ell_1 A$. An arrow from f to $f' : \ell_1 A' \to \ell_1 A'$ is a pair of basic embeddings $\langle e, e \rangle$ commuting with f and f', that is, $e \circ f = f' \circ e$. Furthermore, we may require that $A \subseteq A'$ and $\widehat{e}$ is the inclusion $A \subseteq A'$.

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We say that $\mathcal{B}$ is cofinal in $\mathcal{S}$ if for every object $X \in \mathcal{S}$ there exists an object $Y \in \mathcal{B}$ such that $\mathcal{S}(X, Y) \neq \emptyset$. Let us recall that category $\mathcal{B}$ is dominating in $\mathcal{S}$ if the family of objects $\mathrm{Dom}(\mathcal{B})$ is cofinal in $\mathcal{B}$ and moreover for every $A \in \mathrm{Dom}(\mathcal{B})$ and for every arrow $f : A \to X$ in $\mathcal{S}$ there exists an arrow g in $\mathcal{S}$ such that $g \circ f \in \mathcal{B}$, where $\mathrm{Dom}(\mathcal{B}) = \{\mathrm{dom}(f) : f \in \mathcal{B}\}$ and $\mathcal{B}$ denote family of arrow.

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$\mathcal{S}$, as well as $\mathcal{B}$, has the amalgamation property.

We say that $\mathcal{B}$ is cofinal in $\mathcal{S}$ if for every object $X \in \mathcal{S}$ there exists an object $Y \in \mathcal{B}$ such that $\mathcal{S}(X, Y) \neq \emptyset$. Let us recall that category $\mathcal{B}$ is dominating in $\mathcal{S}$ if the family of objects $\mathrm{Dom}(\mathcal{B})$ is cofinal in $\mathcal{B}$ and moreover for every $A \in \mathrm{Dom}(\mathcal{B})$ and for every arrow $f : A \to X$ in $\mathcal{S}$ there exists an arrow g in $\mathcal{S}$ such that $g \circ f \in \mathcal{B}$, where
$\mathrm{Dom}(\mathcal{B}) = \{\mathrm{dom}(f) : f \in \mathcal{B}\}$ and $\mathcal{B}$ denote family of arrow.

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In fact, $\mathcal{S}$ satisfies the axioms of a Fraïssé category, except that it is uncountable. Namely, there are obviously continuum many non-expansive operators on $\ell_1 F$, whenever $|F| > 1$. Denote by $\mathcal{S}_{\mathbb{Q}}$ and $\mathcal{B}_{\mathbb{Q}}$ the rational variants of $\mathcal{S}$ and $\mathcal{B}$, respectively. In both cases we restrict to rational operators, i.e., operators mapping each of the basic vectors to a rational combination of basic vectors. Since there are only finitely many basic embeddings between finite-dimensional spaces of the form $\ell_1 S$, our rational categories are essentially countable. Now, from the general Fraïssé theory we know that there exists a sequence in $\mathcal{B}_{\mathbb{Q}}$ that is Fraïssé in $\mathcal{S}_{\mathbb{Q}}$. Its limit is a non-expansive operator $\Omega : \ell_1 \to \ell_1$. This is because ℓ_1 is, up to isometry, the unique separable infinite-dimensional space of the form $\ell_1 S$.

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