UNIVERSITY OF LODZ

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
University of Lodz

A visit to the family of porous sets

Jacek Hejduk and Piotr Nowakowski

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Let $E \subset \mathbb{R}$. The porosity of a set E at a given point $x \in \mathbb{R}$ is a number

$$p(E,x) = \limsup_{h \to 0^+} \frac{\gamma(E,(x-h,x+h))}{h},$$

where

$\gamma(E,(x-h,x+h)) = \sup\{|J| \colon J \subset (x-h,x+h) \setminus E, J \text{ - an open interval}\}.$

A set E is porous at a point $x \in \mathbb{R}$ if $p(E,x) > 0$. A set E is porous if it is porous at every point $x \in E$.

Putting for every set $A \subset \mathbb{R}$

$$\Phi(A) = \{x \in \mathbb{R} \colon p(A',x) > 0\}$$

we get an operator satisfying the conditions:

1. $\Phi(\emptyset) = \Phi(\emptyset)$, $\Phi(\mathbb{R}) = \mathbb{R}$;
2. $\forall_{A\text{- finite set}} \Phi(A) = \emptyset$
3. $\forall_{A,B \subset \mathbb{R}}\ A \subset B \Rightarrow \Phi(A) \subset \Phi(B)$;
4. $\forall_{I\text{-interval}}\ I \subset \Phi(I)$;
5. $\forall_{A \in \tau_{nat}}\ A \subset \Phi(A)$;
6. $\forall_{A \in \tau_{nat}} \forall_{B \subset \mathbb{R}}\ (B \subset \Phi(B) \Rightarrow A \cap B \subset \Phi(A \cap B)$.

Operator Φ is not comparable with the operator Φ_d and Φ_I.

$$\Phi_d(\mathbb{R} \setminus \mathbb{Q}) = \Phi_I(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R},$$

but

$$\Phi(\mathbb{R} \setminus \mathbb{Q}) = \emptyset.$$

and also

$$\Phi_d([a, b]) = \Phi_I([a, b]) = (a, b),$$

but

$$\Phi([a, b]) = [a, b].$$

Theorem 1

A family

$$\tau = \{A \subset \mathbb{R} \colon A \subset \Phi(A)\}$$

is a strong generalized topology, which means that $\emptyset, \mathbb{R} \in \tau$ *and* τ *is closed with respect to the arbitrary unions. Moreover,* $\tau_{nat} \subsetneq \tau$.

The notion of a generalized topology was invented indepedently by many mathematicians,e.g.: E.H. Moore , A. Apert, T.S. Motzkin, F.W. Levi, S. Lugojan, A.S. Mashhour, A.A. Allam, F.S. Mahmooud, F.H. Khedr, and Á. Császár.

Example 1

$$\mathcal{T}^* = \{A \subset \mathbb{R} : \forall_{x\in A} \lim_{h\to 0} \frac{\mu^*(A\cap[x-h,x+h])}{2h} = 1\}$$

(J. Hejduk, A. Loranty *On strong generalized topology with respect to the outer Lebesgue measure*, Acta Math. Hungar. **163**(1) (2021))

Example 2

$$\mathcal{T}^+ = \{A \in \mathcal{L} : \forall_{x\in A} \limsup_{h\to 0} \frac{\mu(A\cap[x-h,x+h])}{2h} > 0\}$$

(J. Hejduk, R. Wiertelak, W. Wilczyński *On the family of measurable sets having the upper positive density*, submitted.)

The definition $p(A', x) > 0$ for $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ implies that A contains an interval. Therefore, for every $A \in \tau \setminus \{\emptyset\}$ we have $int_{\tau_{nat}} A \neq \emptyset$.

Proposition 2

The strong generalized topology τ and τ_{nat} are similar, which means that

$$\forall_{A \subset \mathbb{R}} \, (int_{\tau} A \neq \emptyset \Leftrightarrow int_{\tau_{nat}} A \neq \emptyset).$$

Example 3

Let us define

$$A = [0,1) \setminus \bigcup_{n\in\mathbb{N}} \left\{ \frac{1}{2^n} \right\}, \quad B = [0,1) \setminus \bigcup_{n\in\mathbb{N}} \left\{ \frac{1}{n} \right\}.$$

Then $A \in \tau$ but $B \notin \tau$, because $0 \in B \setminus \Phi(B)$.

Simultaneously $A \triangle B \in \mathcal{J}$, where $\mathcal{J}$ is an arbitrary σ-ideal in $\mathbb{R}$ containing all singletons. However, $\Phi(A) \neq \Phi(B)$.

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Remark 3

If $A \triangle B$ is finite then $\Phi(A) = \Phi(B)$.

Proposition 4

$$\forall_{A\subset\mathbb{R}}\ \Phi(A)\setminus A\in\mathcal{J}_P,$$

where $\mathcal{J}_P$ *is the family of porous sets. Consequently,*

$$\Phi(A)\setminus A\in\mathbb{L}\cap\mathcal{N}_{\tau_{nat}},$$

so it is Lebesgue null set and nowhere dense set.

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The opposite property that $A\setminus\Phi(A)\in\mathcal{J}_P$ does not hold, because $\Phi(\mathbb{R}\setminus\mathbb{Q})=\emptyset$.

Theorem 5

A set $U \in \tau \setminus \{\emptyset\}$ *if and only if* $U = \bigcup_{n\in\mathbb{N}} I_n \cup A$*, where* (I_n) *is a sequence of non-trivial components,* $A \cap \bigcup_{n\in\mathbb{N}} I_n = \emptyset$ *and for every* $x \in A$ *there exists a subsequence* (I_{t_n}) *such that* $\lim_{n\to\infty} d(x, I_{t_n}) = 0$ *and*

$$\limsup_{n\to\infty} \frac{|I_{t_n}|}{|I_{t_n}| + d(x, I_{t_n})} > 0.$$

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A set $U \in \tau \setminus \{\emptyset\}$ *if and only if* $U = \bigcup_{n\in\mathbb{N}} I_n \cup A$, *where* (I_n) *is a sequence of non-trivial components,* $A \cap \bigcup_{n\in\mathbb{N}} I_n = \emptyset$ *and for every* $x \in A$ *there exists a subsequence* (I_{t_n}) *such that* $\lim_{n\to\infty} d(x, I_{t_n}) = 0$ *and*

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Corollary 6

Let (I_n) *be a sequence of non-trivial intervals such that* $x \notin \bigcup_{n\in\mathbb{N}} I_n$ *and* $\lim_{n\to\infty} d(x, I_{t_n}) = 0$. *Then* $U = \bigcup_{n\in\mathbb{N}} I_n \cup \{x\} \in \tau$ *if and only if there exists a subsequence* (I_{t_n}) *such that*

$$\limsup_{n\to\infty} \frac{|I_{t_n}|}{|I_{t_n}| + d(x, I_{t_n})} > 0.$$

Corollary 7

Let (x_n) be a decreasing sequence converging to x. Then a set $U = \bigcup_{n\in\mathbb{N}}(x_{n+1}, x_n) \cup \{x\} \in \tau$ if and only if

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Corollary 7

Let (x_n) be a decreasing sequence converging to x. Then a set $U = \bigcup_{n\in\mathbb{N}}(x_{n+1}, x_n) \cup \{x\} \in \tau$ if and only if

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Theorem 8

For every set $A \subset \mathbb{R}$:

i) $int_\tau(A) = A \cap \Phi(A)$

ii) $cl_\tau(A) = A \cup (X \setminus \Phi(X \setminus A))$.

Let (X, Γ) be a generalized topology.

Definition 4

A set $A \subset X$ is

- Γ-nowhere dense if $int_\Gamma(cl_\Gamma(A)) = \emptyset$,
- Γ-strong nowhere dense if for every set $U \in \Gamma \setminus \{\emptyset\}$ there exists a set $V \in \Gamma \setminus \{\emptyset\}$ such that $V \subset U$ and $V \cap A = \emptyset$,
- Γ-first category set if it is a countable union of Γ-nowhere dense sets,
- Γ- strong first category set if it is a countable union of Γ-strong nowhere dense sets.

Theorem 9

For every set $A \subset \mathbb{R}$ the following conditions are equivalent

i) *A is τ-nowhere dense,*

ii) *A is τ-strong nowhere dense,*

iii) *A is τ_{nat}-nowhere dense,*

- Separability

Theorem 10

$(\mathbb{R}, \tau)$ *is separable.*

- The first countability

Theorem 11

$(\mathbb{R}, \tau)$ *is not the first countable.*

- The compactness

Theorem 12

$A \subset \mathbb{R}$ is τ-compact if and only if it is finite.

- The axioms of separations

Theorem 13

$(\mathbb{R}, \tau)$ *is a Hausdorff space.*

Theorem 14

$(\mathbb{R}, \tau)$ *is a normal space.*

- Connectivity

Theorem 15

$(\mathbb{R}, \tau)$ *is a extremally disconnected.*

Theorem 16

The topology generated by τ is equal to $\mathcal{P}(\mathbb{R})$.

Let (X, Γ) be a generalized topology. It is known:

Theorem 17

A family

$$\mathcal{T}_\Gamma = \{A \in \Gamma : \forall_{B \in \Gamma}\ A \cap B \in \Gamma\}$$

is a topology included in Γ*.*

(In the case of example 2 $\mathcal{T}_\Gamma = \mathcal{T}_d$ -density topology) So that

$$\mathcal{T}_\tau = \{A \in \tau : \forall_{B \in \tau}\ A \cap B \in \tau\}$$

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$$\tau_{nat} \subset \mathcal{T}_\tau.$$

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$$\tau_{nat} \subset \mathcal{T}_\tau$$

Question 1

$$\tau_{nat} = \mathcal{T}_\tau ?$$

Continuity

Let $f : \langle \mathbb{R}, \tau \rangle \to \langle \mathbb{R}, \tau_{nat} \rangle$.

Definition 5

We shall say that f is τ-continuous at $x_0 \in \mathbb{R}$ if for every $W \in \tau_{nat}$, $f(x) \in W$ there exists $V \in \tau$ such that $x_0 \in V$ and $f(V) \subset W$.

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If $f : \langle \mathbb{R}, \tau \rangle \to \langle \mathbb{R}, \tau_{nat} \rangle$ is τ-approximately continuous at $x_0 \in \mathbb{R}$ then f is τ-continuous.

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Question 2

Is the inverse property true?

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