On the porosity of Baire class function

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S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math. **19** (1932), 184–197.

Definition (S. Kempisty, 1932)

A function f is quasi-continuous at a point x if for every neighbourhood U of x and for every neighbourhood V of $f(x)$ there exists a non-empty open set $G \subset U$ such that $f(G) \subset V$.

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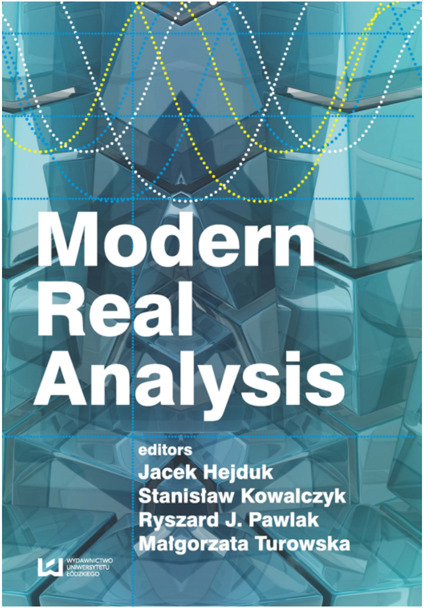
$\mathcal{Q}$ - the family of all quasi-continuous functions

To which class do quasi-continuous functions belong?



Jan Borsik "Quasicontinuous functions with small set of discontinuity points"

S. Marcus in [49] proved that the notions of neighborly and quasicontinuous functions are equivalent and he developed further properties of quasicontinuous functions. He showed that quasi-continuous functions need not be (Lebesgue) measurable and for each countable ordinal α there is a quasicontinuous function in the Baire class $\alpha + 1$ which does not belong to Baire class α.

S. Marcus, *Sur les fonctions quasicontinues au sens de S. Kempisty*, Coll. Math. 8 (1961), 47–53.

L. Bukovský, *The structure of the Real line*, Monografie Matematyczne (New Series), vol. 71. Birkhäuser, Basel (2011).

- $\Sigma_0^0 = \Pi_0^0 = \Delta_0^0$ the set of all clopen subsets of $(\mathbb{R}, \tau_e)$
- Σ_1^0 the set of all open subsets of $(\mathbb{R}, \tau_e)$
- Π_1^0 the set of all closed subsets of $(\mathbb{R}, \tau_e)$
- $\Sigma_\alpha^0 = \left\{ \bigcup_n A_n : \; A_n \in \bigcup_{\beta<\alpha} \Pi_\beta^0, \; n \in \omega \right\}$
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Σ_2^0 - the family of F_σ sets
Π_2^0 - the family of G_δ sets.

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- $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$;
- $f \in \mathcal{B}_\alpha$ iff for each $V \in \tau_e$ holds $f^{-1}(V) \in \Sigma^0_{\alpha+1}$

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(xi) *Quel que soit l'ordinal α de première ou seconde classe transfinie, il existe une fonction quasicontinue dans I et de classe α de Baire effective dans I.*

En effet, $P \subset I$ étant un ensemble parfait rare et de mesure positive, désignons par N l'ensemble des extrémités de tous les intervalles fermés $I_1, I_2, \ldots$ contigus à P, supposés numérotés de façon que chaque voisinage d'un point de $P-N$ contienne un intervalle à indice pair et un autre à indice impair. Considérons un ensemble arbitraire $R \subset P-N$ et posons pour $n = 1, 2, \ldots$

$$f(x) = \begin{cases} 0 \text{ lorsque } x \in R \text{ ou bien } x \in I_{2n-1}, \\ 1 \text{ lorsque } x \in P-N-R \text{ ou bien } x \in I_{2n}. \end{cases}$$

La fonction f est évidemment quasicontinue dans I. Si l'ensemble R n'est pas mesurable au sens de Lebesgue (ce qui est possible, P étant supposé de mesure positive), la fonction f n'est pas mesurable non plus, ce qui démontre (x).

Mais on peut choisir R mesurable et tel qu'il soit d'une classe borelienne α donnée d'avance, donc que la fonction f appartienne à la classe α de Baire. On a ainsi (xi).

L. Zajiček, *On σ-porous sets in abstract spaces*, Abstr. Appl. Anal. **5** (2005), 509–534.

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The set $M \subset X$ is

1. porous if $p(M, x) > 0$ for each $x \in M$
2. strongly porous if $p(M, x) = 1$ for each $x \in M$

$f, g \in \mathcal{B}$

$$\boxed{\rho(f,g) = min\{1, sup\{| f(t) - g(t) |: t \in \mathbb{R}\}\}}$$

Lemma

A set $F \in \Delta^0_{\alpha+1}$ *iff* $\chi_F \in \mathcal{B}_\alpha$.

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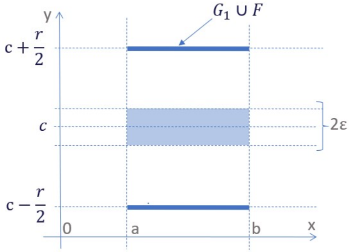

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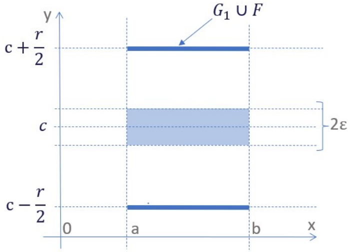

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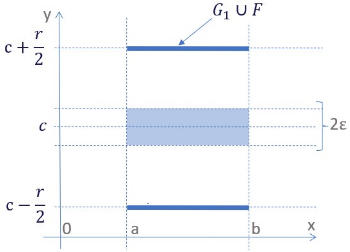

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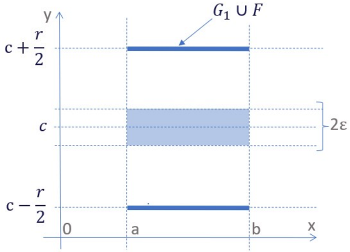

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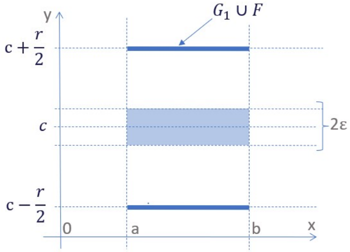

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$$p\left(\mathcal{Q} \cap \mathcal{B}_\alpha, f\right) = 2 \limsup_{r \to 0^+} \frac{\gamma(f, r, \mathcal{Q} \cap \mathcal{B}_\alpha)}{r} = 1$$

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1. $\mathcal{C} = \mathcal{Q} \cap \mathcal{C} \subset \mathcal{C}_q \cap \mathcal{C} \subset \mathcal{C}$, for $\alpha = 0$
2. $\mathcal{Q} \cap \mathcal{B}_1 \subset \mathcal{C}_q \cap \mathcal{B}_1 = \mathcal{B}_1 \subset \mathcal{C}_q$, for $\alpha = 1$

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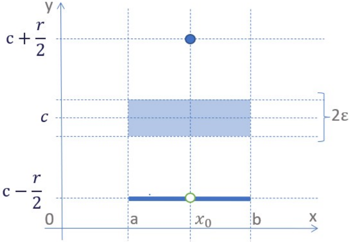

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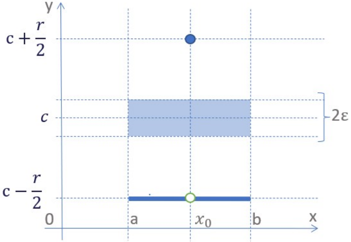

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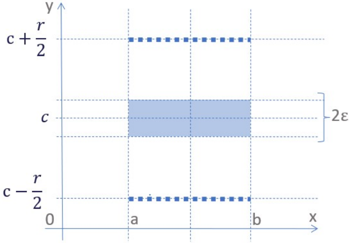

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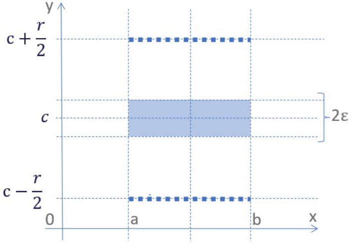

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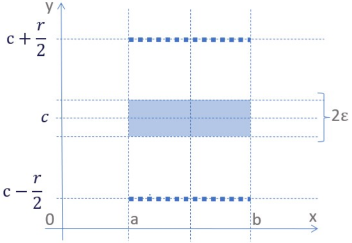

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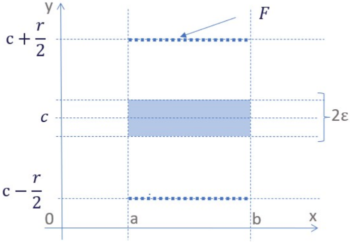

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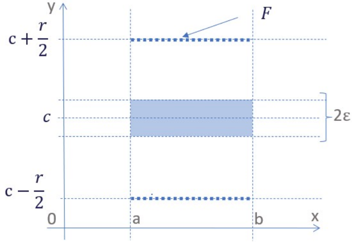

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For each ordinal number $\alpha < \omega_1$ the set $\mathcal{B}_\alpha$ is strongly porous in $(\mathcal{B}_{\alpha+1}, \varrho)$.

Corollary

For each ordinal numbers $\alpha < \beta < \omega_1$

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2. $\mathcal{B}_\alpha \cap \mathcal{Q}$ *is strongly porous in* $(\mathcal{B}_\beta \cap \mathcal{Q}, \varrho)$*;*
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