On a problem of Rudin concerning Baire classification of separately continuous functions

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Outline of the talk

- Rudin Theorem and the Problem
- Generalizations of Rudin Theorem – I
- Generalizations of Rudin Theorem – II
- Generalizations of Rudin Theorem – III
- Three questions of Volodymyr Maslyuchenko
- Sorgenfrey line and separately continuous functions
- Pictures :)

Separately continuous functions

Definition

A function $f : X \times Y \to Z$ is **separately continuous** if

$$\forall x \in X \quad f^x : Y \to Z \text{ is continuous,}$$
$$\forall y \in Y \quad f_y : X \to Z \text{ is continuous,}$$

where $f^x(y) = f_y(x) = f(x, y)$

Schwartz function

$$\varphi(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & x^2 + y^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$CC(X \times Y, Z)$

Baire classification of CC-functions

Lebesgue Theorem, 1898

Every separately continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a pointwise limit of a sequence of continuous functions.

A function $f : X \to Y$ belongs to the **first Baire class**, if it is a pointwise limit of a sequence of continuous functions.
$\mathrm{B}_1(X, Y)$

Baire classification of CC-functions

Lebesgue Theorem, 1898

Every separately continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a pointwise limit of a sequence of continuous functions.

Lebesgue's construction:

$$f_n(x,y) = n\left(f\left(\tfrac{k}{n}, y\right)\left(\tfrac{k+1}{n} - x\right) + f\left(\tfrac{k+1}{n}, y\right)\left(x - \tfrac{k}{n}\right)\right),$$
$$(x,y) \in [\tfrac{k}{n}, \tfrac{k+1}{n}] \times \mathbb{R},\ k \in \mathbb{Z}.$$

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Baire classification of CC-functions

$C\overline{C}(X \times Y, Z)$

Rudin Theorem, 1981

Let X be a metrizable space, Y be a topological space and Z be a locally convex topological vector space. Then $C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z)$.

<u>Rudin's construction:</u>

- $(\varphi_{i,n} : i \in I_n)$ is a locally finite partition of unity on X
- $\mathrm{diam}(\mathrm{supp}\varphi_{i,n}) \to 0$ for $n \to \infty$
- $x_{i,n} \in \mathrm{supp}\varphi_{i,n}$ and $f^{x_{i,n}} : Y \to Z$ is continuous
- $f_n(x, y) = \sum\limits_{i \in I_n} \varphi_{i,n}(x) f(x_{i,n}, y)$

Problem of Rudin

Do there exist a metrizable space X, a topological space Y, a topological vector space (or, more general, an equiconnected space) Z and a separately continuous mapping $f : X \times Y \to Z$ which is not Baire 1?

Generalizations of Rudin Theorem – I

A. Kalancha and V. Maslyuchenko, 2003

Let X be a metrizable space with $\dim X < \infty$, Y be a topological space and Z be a topological vector space. Then $C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z)$.

Corollary

Let Y be a topological space and Z be a topological vector space. Then $C\overline{C}(\mathbb{R}^n \times Y, Z) \subseteq B_1(\mathbb{R}^n \times Y, Z)$.

Generalizations of Rudin Theorem – II

Definition

A topological space X is semi-stratifiable, if there exists a sequence of open sets $(U_{x,n} : x \in X)$ in X such that

$$\{x\} = \bigcap_{n \geq 1} U_{x,n}$$

$$\text{if } \forall n \;\; x \in U_{x_n,n}, \;\; \text{then } \; x_n \to x$$

If, moreover, for every closed set $F \subseteq X$ we get the equality

$$F = \bigcap_{n=1}^{\infty} \overline{\bigcup_{x \in F} U_{x,n}},$$

the space X is called stratifiable.

Generalizations of Rudin Theorem – II

- O. Sobchuk (2002) : PP-spaces
- T. Banakh (2002) : metrically quarter-stratifiable spaces

Generalizations of Rudin Theorem – II

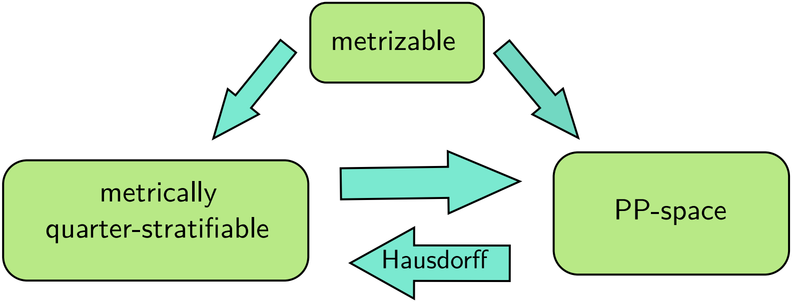
A topological T_1-space $(X, \mathscr{S})$ is called

- (metrically) quarter-stratifiable, if there exist a weaker metrizable topology τ, a sequence of τ-open coverings $(U_{i,n} : i \in I_n)$ of X and a sequence $(x_{i,n} : i \in I_n)$ of families of points in X such that

$$\text{if } \forall n \;\; x \in U_{i_n,n}, \;\; \text{then } \; x_{i_n,n} \to x \qquad (1)$$

- a PP-space, if there exists a sequence of locally finite coverings $(U_{i,n} : i \in I_n)$ of X by cozero sets and a sequence $((x_{i,n} : i \in I_n))_{n=1}^{\infty}$ of families of points in X such that (1) holds.

Generalizations of Rudin Theorem – II



Equiconnected spaces

An equiconnected space is a pair (X, λ) consisting of a topological space X and a continuous map $\lambda : X \times X \times [0,1] \to X$ satisfying the following conditions

(Λ_1) $\lambda(x, y, 0) = x$,

(Λ_2) $\lambda(x, y, 1) = y$,

(Λ_3) $\lambda(x, x, t) = x$

for all $x, y \in X$ and $t \in [0,1]$.

TVS $\rightarrow$ equiconnected $\rightarrow$ contractible and locally contractible

Equiconnected spaces

Let (X, λ) be an equiconnected space and $\emptyset \neq A \subseteq X$. We define

$$\lambda^0(A) = A,$$
$$\lambda^n(A) = \lambda(\lambda^{n-1}(A) \times A \times [0,1]) \text{ for } n \in \mathbb{N},$$
$$\lambda^\infty(A) = \bigcup_{n=1}^{\infty} \lambda^n(A).$$

Definition

We say that an equiconnected space (X, λ) is locally convex, if for any $x \in X$ and any neighborhood U of x there exists a neighborhood V of x such that $\lambda^\infty(V) \subseteq U$.

Generalizations of Rudin Theorem – II

Theorem

Every separately continuous function $f : X \times Y \to Z$ *is Baire 1, if*

- X *is a PP-space,* Y *is a topological space and* Z *is a locally convex topological vector space (O. Sobchuk);*
- X *is a metrically quarter-stratifiable paracompact and strongly countable-dimensional space,* Y *is a topological space and* Z *is an equiconnected space (T. Banakh);*
- X *is a metrically quarter-stratifiable,* Y *is a topological space and* Z *is an equiconnected locally convex space (T. Banakh).*

An example of $C\overline{C}$-function which is not Baire 1

Let $X = \{0\} \cup \bigcup_{n=1}^{\infty} X_n$, where $X_n = \{\frac{1}{n}\} \cup \bigcup_{m=n^2}^{\infty} \{\frac{1}{n} + \frac{1}{m}\}$. We define a topology on X in the following way. All points of the form $\frac{1}{n} + \frac{1}{m}$ will be isolated points of X. The base of neighborhoods of a point $\frac{1}{n}$ are the sets of the form $X_n \setminus \bigcup_{m=n^2}^{k} \{\frac{1}{n} + \frac{1}{m}\}$,
$k = n^2, n^2 + 1, \ldots$.
As neighborhoods of 0 we take all the sets obtained from X by removing a finite number of X_n's and a finite number of points of the form $\{\frac{1}{n} + \frac{1}{m}\}$ in all the remaining X_n's.
The space X with this topology is called Arens fan.
X is σ-metrizable and paracompact $\implies$ X is metrically quarter-stratifiable

An example of $C\overline{C}$-function which is not Baire 1

Example

There exists a function $f \in C\overline{C}(X \times \mathbb{R}, \mathbb{R})$, which is not a pointwise limit of a sequence of separately continuous functions $f_n : X \times \mathbb{R} \to \mathbb{R}$.

Let $g : \mathbb{R} \to \mathbb{R}$, $g = \chi_{\mathbb{Q}}$. Then

$$g(y) = \lim_{n\to\infty} g_n(y) = \lim_{n\to\infty} \lim_{m\to\infty} g_{n,m}(y)$$

for all $y \in \mathbb{R}$.
Let $x_0 = 0$, $x_n = \frac{1}{n}$, $x_{nm} = \frac{1}{n} + \frac{1}{m}$, where $m \geq n^2$, and consider the function $f : X \times \mathbb{R} \to \mathbb{R}$,

$$f(x,y) = \begin{cases} g(y), & x = x_0, \\ g_n(y), & x = x_n, \\ g_{nm}(y), & x = x_{nm}. \end{cases}$$

Strong PP-spaces

Definition

A topological space X is a strong PP-space if for any dense set $D \subseteq X$ there exist a sequence $(U_{i,n} : i \in I_n)_{n=1}^{\infty}$ of locally finite coverings of X by cozero sets and a sequence $(x_{i,n} : i \in I_n)$, $x_{i,n} \in D$, such that

$$\text{if } \ x \in U_{i_n,n}, \ \text{ then } \ x_{i_n,n} \to x$$

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  3. Take $x_{i,n} \in D \cap U_{i,n} \quad \forall i \in I_n$
- Every σ-metrizable paracompact space is metrically quarter stratifiable

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Examples of (strong) PP-spaces

- $\mathbb{R}^\infty$ is a σ-metrizable paracompact space, but is not a strong PP-space:

$$\mathbb{R}^\infty = \{(\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots) : \xi_i \in \mathbb{R}\}.$$

Denote by E the set of all sequences $e = (\varepsilon_n)_{n=1}^\infty$ of positive reals ε_n and let

$$U_e = \{x = (\xi_n)_{n=1}^\infty \in \mathbb{R}^\infty : (\forall n \in \mathbb{N})(|\xi_n| \leq \varepsilon_n)\}.$$

We consider on $\mathbb{R}^\infty$ the topology in which the system

$$\mathcal{U}_0 = \{U_e : e \in E\}$$

forms the base of neighborhoods of zero.

Examples of (strong) PP-spaces

- $\mathbb{R}^\infty$ is a σ-metrizable paracompact space, but is not a strong PP-space:
  Let $A_n = \{(\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) : |\xi_k| \le \frac{1}{n} \;\; \forall k \le n\}$,

  $$D = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{m} (\mathbb{R}^\infty \setminus A_n).$$

  Then $\overline{D} = \mathbb{R}^\infty$, but there is no sequence in D which is convergent to $x = (0, 0, 0, \ldots)$ in $\mathbb{R}^\infty$.

Examples of (strong) PP-spaces

- Sorgenfrey line $\mathbb{S}$ is not metrizable, not semi-stratifiable and is a strong PP-space:

Examples of (strong) PP-spaces

- Sorgenfrey line $\mathbb{S}$ is not metrizable, not semi-stratifiable and is a strong PP-space:
  Fix a dense set $D \subseteq \mathbb{S}$ and let

$$U_{i,n} = \left[\tfrac{i-1}{n}, \tfrac{i}{n}\right), i \in \mathbb{Z}, n \in \mathbb{N},$$

$$x_{i,n} \in D \cap \left[\tfrac{i}{n}, \tfrac{i+1}{n}\right)$$

Generalizations of Rudin Theorem – II

Definition

A topological space X is strongly countably dimensional, if $X = \bigcup_{n=1}^{\infty} X_n$, X_n is closed and $\dim X_n < n$ for every $n \in \mathbb{N}$.

Theorem [K., Mykhaylyuk and V. Maslyuchenko]

Let X be a strong PP-space, Y a topological space, Z an equiconnected space. If one of the following conditions holds

- X is Hausdorff paracompact strongly countably dimensional,
- Z a locally convex,

then

$$C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z).$$

Generalizations of Rudin Theorem – II

Corollary

Let X be a strongly countably dimensional metrizable space, Y a topological space, Z a topological vector space. Then

$$C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z).$$

Generalizations of Rudin Theorem – III

Question (T. Banakh, 2003)

Let X, Y, Z be a metrizable spaces, X and Y be compact and Z be a topological vector space. Does every separately continuous function $f : X \times Y \to Z$ belong to the first Baire class?

Generalizations of Rudin Theorem – III

- If X, Y and Z are metrizable, then every separately continuous function $f : X \times Y \to Z$ is Borel 1 (Montgomery, Kuratowski);

Generalizations of Rudin Theorem – III

- If X, Y and Z are metrizable, then every separately continuous function $f : X \times Y \to Z$ is Borel 1 (Montgomery, Kuratowski);
- If T is metrizable, then every Borel 1 function $f : T \to \mathbb{R}$ is Baire 1 (Lebesgue, Hausdorff).

Generalizations of Rudin Theorem – III

Theorem [K., 2023]

Let X be a strong PP-space, Y be a topological space, Z be a metrizable space and $f \in C\overline{C}(X \times Y, Z)$. Then f is Borel 1 and functionally σ-discrete.

Theorem [K. 2017]

Let T be a topological space and Z be a metrizable connected and locally arcwise connected space. Then every Borel 1 functionally σ-discrete map $f : T \to Z$ is Baire 1.

Generalizations of Rudin Theorem – III

Theorem [K., 2023]

Let X be a strong PP-space, Y be a topological space, Z be a metrizable connected and locally arcwise connected space. Then

$$C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z).$$

The first question of Volodymyr Maslyuchenko

Question 1 by Volodymyr Maslyuchenko

Do there exist an arcwise connected and locally arcwise connected space Z and a separately continuous function $f : \mathbb{R} \times \mathbb{R} \to Z$ which is not Baire 1?

The first question of Volodymyr Maslyuchenko

Let X, Y be topological spaces. A set $G \subseteq X \times Y$ is open in **cross topology** γ on $X \times Y$, if every point $(x, y) \in G$ is contained in G with a "cross" $(U \times \{y\}) \cup (\{x\} \times V)$ for some open sets $U \subseteq X$ and $V \subseteq Y$.

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$X \otimes Y$

The first question of Volodymyr Maslyuchenko

If X, Y, Z are topological spaces, then:

- $f : X \times Y \to Z$ is separately continuous $\Leftrightarrow$ f is continuous with respect to γ;

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If X, Y, Z are topological spaces, then:

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- $f : X \times Y \to Z$ is separately continuous $\Leftrightarrow$ f is continuous with respect to γ;
- $X \otimes Y$ is T_2 if X and Y are T_2;
- $\mathbb{R} \otimes \mathbb{R}$ is not T_3 (Sierpiński);
- $\mathbb{R} \otimes \mathbb{R}$ is arcwise connected and locally arcwise connected.

The first question of Volodymyr Maslyuchenko

Theorem (K. and Mykhaylyuk, 2013)

The identity map $\mathrm{id} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$ is separately continuous and is not Baire 1.

The second question of Volodymyr Maslyuchenko

Let us consider on ℓ_p $(0 < p < 1)$ a topology, generated by pre-norms s-pre-norms

$$|x|_y = \sum_{k=1}^{\infty} |\xi_k \eta_k|^s$$

for $s < p$, where

- $\ell_q^+ = \{y = (\eta_k)_{k=1}^{\infty} \in \ell_q : \eta_k \geq 0 \quad \forall\, k\}$,
- $x = (\xi_k)_{k=1}^{\infty} \in \ell_p$, $y = (\eta_k)_{k=1}^{\infty} \in \ell_q^+$,
- $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{s}$.

The sets

$$U_y = \{x \in \ell_p : |x|_y \leq 1\}, \quad \text{where } y \in \ell_q^+$$

form a base of neighborhoods of zero. We denote this topology by $\varkappa_s$.

The second question of Volodymyr Maslyuchenko

$(\ell_p, \varkappa_s)$ is not locally convex, but is σ-metrizable arcwise connected and locally arcwise connected

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Question 2 by Volodymyr Maslyuchenko

Does every separately continuous function $f : \mathbb{R} \times \mathbb{R} \to (\ell_p, \varkappa_s)$ belong to the first Baire class?

The second question of Volodymyr Maslyuchenko

Theorem (K.)

$C\overline{C}(\mathbb{R} \times \mathbb{R}, (\ell_p, \varkappa_s)) \subseteq B_1(\mathbb{R} \times \mathbb{R}, (\ell_p, \varkappa_s))$.

The third question of Volodymyr Maslyuchenko

Question 3 by Volodymyr Maslyuchenko

Let X, Y be metrizable compact spaces and $f : X \times Y \to \mathbb{R}$ be a separately continuous function. Then f is Baire 1.
Does there exist a sequence of continuous functions $f_n : X \times Y \to \mathbb{R}$ which is layerwisely uniformly convergent to f?

$f_n \overset{\textit{l.u.}}{\rightrightarrows} f$, if $\forall (x, y) \in X \times Y$ we have

- $f_n^x \rightrightarrows f^x$ on $\{x\} \times Y$ and
- $(f_n)_y \rightrightarrows f_y$ on $X \times \{y\}$

The third question of Volodymyr Maslyuchenko

The positive answer was given in [MV] for $f : [0,1]^2 \to \mathbb{R}$ in two particular cases:

- if the set $D(f)$ of all points of discontinuity of f has at most countable projection E on the first axis;
- if the restriction $f|_{E\times[0,1]}$ is continuous.

V. Maslyuchenko, H. Voloshyn, *A topologization of the space of separately continuous functions*, Carpathian Math. J. **5** (2) (2013).

The third question of Volodymyr Maslyuchenko

The positive answer was given by Taras Banakh in the case

- if X, Y are separable metrizable spaces, Z is a metrizable topological group and the image $f(X \times Y)$ is zero-dimensional.

T. Banakh, *On the sequential closure of the set of continuous functions in the space of separately continuous functions*, Buk. Math. J. **3** (2) (2015).

The third question of Volodymyr Maslyuchenko

Theorem [K. and Volodymyr Mykhalyuk]

Let X be a stratifiable hereditarily Baire space, Y be a compact space, Z be a metrizable space and $f : X \times Y \to Z$ be a separately continuous map. If one of the following conditions holds

- Z is a locally convex equiconnected space,
- $\dim X < \infty$ and Z is equiconnected,
- $\dim X = 0$,

then there exists a sequence of continuous maps $f_n : X \times Y \to Z$ which is convergent to f layerwisely uniformly.

Outline of the proof

Step 1. Application of Namioka property

A map $f : X \times Y \to Z$ has *the Namioka property*, if there exists a dense G_δ subset A of X such that $A \times Y \subseteq C(f)$.

A space X is said to be

- *a Namioka space*, if for any compact space Y and metrizable space Z every separately continuous function $f : X \times Y \to Z$ has the Namioka property;
- *a hereditarily Namioka space*, if every closed subset $F \subseteq X$ is a Namioka space.

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Theorem

Every quater-stratifiable Baire space X is Namioka.

Theorem

Every semi-stratifiable hereditarily Baire space X is hereditarily Namioka.

Outline of the proof

Step 1. Application of Namioka property

$f : X \times Y \to Z$ is separately continuous

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Take a closed set $F \subseteq X$

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Hence, there is a point of continuity $x_0 \in F$ of ψ_F

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Take a closed set $F \subseteq X \implies F$ is a Namioka space
Hence, there is a point of continuity $x_0 \in F$ of $\psi_F \implies \psi$ is fragmented

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Take a closed set $F \subseteq X \implies F$ is a Namioka space
Hence, there is a point of continuity $x_0 \in F$ of $\psi_F \implies \psi$ is fragmented

Definition

We say that $f : X \to Y$ is fragmented, if for every $\varepsilon > 0$ and for every non-empty (closed) set $F \subseteq X$ there exists a point $x \in F$ such that

$$\omega_{f|_F}(x) < \varepsilon.$$

Outline of the proof

Step 2. Space $C(Y, Z)$ is locally convex equiconnected

Theorem

Let Y be a compact space, Z be a metrizable equiconnected space and let $C(Y, Z)$ be the space of all continuous maps with the topology of the uniform convergence. Then $C(Y, Z)$ is a (locally convex) equiconnected space, whenever Z is (locally convex) equiconnected.

Outline of the proof

Step 3. Baire classification of fragmented maps

Theorem

Let X be a semi-stratifiable paracompact space, Z be a metrizable space, $f : X \to Z$ be a fragmented map. If one of the following conditions holds:

1. Z is a locally convex equiconnected space,
2. $\dim X < \infty$ and Z is equiconnected,
3. $\dim X = 0$,

then $f \in \mathrm{B}_1(X, Z)$.

Outline of the proof

Step 3. Baire classification of fragmented maps

$\psi : X \to C(Y, Z)$ is fragmented $\Rightarrow$ ψ is Baire 1

Outline of the proof

Step 3. Baire classification of fragmented maps

$\psi : X \to C(Y, Z)$ is fragmented $\Rightarrow$ ψ is Baire 1
Moreover, there exists a sequence of continuous functions $\psi_n : X \to C(Y, Z)$ such that $\psi_n(x) \to \psi(x)$ in $C(Y, Z)$ and $f_n^x = \psi_n(x)$, $f^x = \psi(x)$

Sorgenfrey line

Theorem (K. and Mykhaylyuk)

Let $\mathbb{S}$ be Sorgenfrey line, Y be a compact space, Z be a metrizable space and let $f : \mathbb{S} \times Y \to Z$ be a separately continuous map. Then there exists a sequence of continuous maps $f_n : \mathbb{S} \times Y \to Z$ such that $f_n \overset{l.u.}{\rightrightarrows} f$.

Sorgenfrey line

William Bade proved that each real-valued continuous function on $\mathbb{S}^2$ belongs to the first Baire class in the topology on $\mathbb{R}^2$. Moreover, Bade noticed that Mrówka obtained the inclusion $\mathrm{C}(\mathbb{S}^{\mathfrak{n}}, \mathbb{R}) \subseteq \mathrm{B}_1(\mathbb{R}^{\mathfrak{n}}, \mathbb{R})$ for every cardinal $\mathfrak{n}$.

W. Bade, *Two properties of the Sorgenfrey plane*, Pacif. J. Math. (1971) 349–354.

S. Mrówka, *Some problems related to N-compact spaces*, unpublished.

Sorgenfrey line

Theorem (K.)

Let Y be a metrizable connected and locally arcwise connected space. Then

$$\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$$

for any set T.

Theorem (K.)

Let Y be a topological vector space. Then the inclusion $\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$ is valid if one of the following conditions hold:

a) $|T| < \aleph_0$,

b) Y is a locally convex space and $|T| \leq \aleph_0$,

c) Y is metrizable.

Open questions

Question 1

Does there exist a completely regular space X which is not a (strong) PP-space and $C\overline{C}(X \times Y, \mathbb{R}) \subseteq B_1(X \times Y, \mathbb{R})$ for any topological space Y?

Question 2

Does the inclusion $\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$ hold for $|T| = \aleph_0$ and any topological vector space Y?

Question 3

Do there exist a metrizable space X, a topological space Y, a topological vector space (or, more general, an equiconnected space) Z and a separately continuous mapping $f : X \times Y \to Z$ which is not Baire 1?

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Pictures :)


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Thank you for the attention!