The continuous type of the Choquet integral representation theorems

Jun Kawabe

Shinshu University

46th Summer Symposium in Real Analysis

University of Lodz, Poland

June 17–21, 2024

- X is a nonempty set.
- $\mathcal{D}$ is a collection of subsets of X containing $\emptyset$.

Definition 1 (nonadditive measure)

A set function $\mu\colon \mathcal{D} \to [0,\infty]$ is called a nonadditive measure on $\mathcal{D}$ if it satisfies

1. $\mu(\emptyset) = 0$,
2. $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{D}$ and $A \subset B$.

Nonadditive measures are widely used in theory and application and have already appeared in many papers under various names: Hausdorff dimension (Hausdorff 1918), lower/upper numerical probability (Koopman 1940), Maharam's submeasure problem (Maharam 1947), capacity (Choquet 1953/54), semivariation (Dunford-Schwartz 1955), quasimeasure (Alexiuk 1968), maxitive measure (Shilkret 1971), participation measure (Tsichritzis 1971), submeasure (Drewnowski 1972, Dobrakov 1974), fuzzy measure (Sugeno 1974), k-triangular set function (Agafanova-Klimkin 1974), game of characteristic function form, distorted measure (Aumann-Shapley 1974), belief/plausibility function (Shafer 1976), possibility measure (Zadeh 1978), pre-measure (Šipoš 1979), necessity measure (Dubois-Prade 1980), approximately additive (Kalton-Roberts 1983), decomposable measure (Weber 1984), Minkowski-Bouligrand dimension (Schroeder 1991), subjective probabilities in decision making, $\cdots\cdots$

What is the Choquet integral representation theorem?

Let X be a nonempty set. Let $\Phi \subset [0,\infty]^X$ and $\Psi \subset \mathbb{R}^X$ with the zero function 0.

- For a monotone functional $I\colon \Phi \to [0,\infty]$ with $I(0) = 0$, there exists a nonadditive measure μ on 2^X such that

$$I(f) = \mathrm{Ch}(\mu, f) := \int_0^\infty \mu(\{f > t\})dt$$

  for every $f \in \Phi$.

- For a monotone functional $I\colon \Psi \to \mathbb{R}$ with $I(0) = 0$, there exists a *finite* nonadditive measure μ on 2^X such that

$$I(f) = \mathrm{Ch}^a(\mu, f) := \int_0^\infty \mu(\{f > t\})dt - \int_{-\infty}^0 \Big\{\mu(X) - \mu(\{f > t\})\Big\}dt$$

  for every $f \in \Psi$.

This type of representation is called the *Choquet integral representation* of I and μ is called a *representing measure* of I

The representing measure μ is generally nonadditive
since no additivity is assumed for I.

We are interested in the continuity of representing measures and will make use of the following continuity of nonadditive measures.

Definition 2

A nonadditive measure $\mu\colon \mathcal{D} \to [0,\infty]$ is called

- inner τ-continuous if $\mu(D_\gamma) \to \mu(D)$ whenever $\{D_\gamma\}_{\gamma\in\Gamma}$ is a net in $\mathcal{D}$, $D \in \mathcal{D}$, and $D_\gamma \uparrow D$.
- outer τ-continuous if $\mu(D_\gamma) \to \mu(D)$ whenever $\{D_\gamma\}_{\gamma\in\Gamma}$ is a net in $\mathcal{D}$, $D \in \mathcal{D}$, and $D_\gamma \downarrow D$.
- conditionally outer τ-continuous if $\mu(D_\gamma) \to \mu(D)$ whenever $\{D_\gamma\}_{\gamma\in\Gamma}$ is a net in $\mathcal{D}$, $D \in \mathcal{D}$, $D_\gamma \downarrow D$, and $\mu(D_{\gamma_0}) < \infty$ for some $\gamma_0 \in \Gamma$.
- inner σ-continuous if $\mu(D_n) \to \mu(D)$ whenever $\{D_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$, $D \in \mathcal{D}$, and $D_n \uparrow D$.
- outer σ-continuous if $\mu(D_n) \to \mu(D)$ whenever $\{D_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$, $D \in \mathcal{D}$, and $D_n \downarrow D$.
- conditionally outer σ-continuous if $\mu(D_n) \to \mu(D)$ whenever $\{D_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$, $D \in \mathcal{D}$, $D_n \downarrow D$, and $\mu(D_{n_0}) < \infty$ for some $n_0 \in \mathbb{N}$.

Zhou (Trans. Amer. Math. Soc. 350 (1998) 1911-1822)

- X is a compact Hausdorff space.
- $C_b(X)$ is the space of all bounded continuous functions on X.
- $I\colon C_b(X) \to \mathbb{R}$ is comonotonically additive and monotone (C.M. for short).

(1) There is a finite nonadditive measure β on 2^X such that β is outer σ-continuous on the collection $\sum_1$ of the compact G_δ-sets and satisfies

$$I(f) = \mathrm{Ch}^a(\beta, f) \tag{1}$$

for every $f \in C_b(X)$.

(2) Conversely, if β is a nonadditive measure on 2^X, then the functional I defined by (1) is a C.M. functional on $C_b(X)$.

Remark 3

We can find another finite representing measure α of I that is inner σ-continuous on the collection $\sum_2$ of the open K_σ-sets. However, it has not yet been discussed whether it is possible to find a single representing measure that is outer σ-continuous on $\sum_1$ and inner σ-continuous on $\sum_2$.

K. (Int. J. Approx. Reason. 54 (2013) 418–426)

- X is a locally compact Hausdorff space.
- $C_{00}(X)$ is the space of all continuous functions on X with compact support.
- $I\colon C_{00}(X) \to \mathbb{R}$ is asymptotically τ-translatable, bounded, and C.M.

(1) There is a finite nonadditive measure α on 2^X such that is α is inner τ-continuous on the collection $\mathcal{G}$ of the open sets and satisfies

$$I(f) = \mathrm{Ch}^a(\alpha, f) \tag{2}$$

for every $f \in C_{00}(X)$.

(2) Conversely, if a nonadditive measure α on 2^X is inner τ-continuous on $\mathcal{G}$, then the functional I defined by (2) is asymptotically τ-translatable, bounded, and C.M. on $C_{00}(X)$.

Remark 4

We can find another finite representing measure β of I that is outer τ-continuous on the collection $\mathcal{K}$ of the compact sets. However, it has not yet been discussed whether it is possible to find a single representing measure that is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$.

The purpose of the study and the benefits of the results

The purpose of the study is to

- find a representing measure that is inner σ-continuous and outer σ-continuous on the respective collections of tractable sets such as the open, the closed, the compact, and the measurable sets. Moreover, in addition to the σ-continuity, discuss the τ-continuity of the representing measures.
- formulate continuous Choquet integral representation theorems in an abstract setting so as to contain as many existing results as possible.
- clarify the relation between regularity and continuity of nonadditive measures.

The benefits of the results are as follows:

- the representing measures are simultaneously inner and outer continuous.
- the respective collections of sets for which the representing measures are inner and outer continuous are larger than those in previous studies.
- the regularity of the representing measures is also clarified.
- it is possible to handle not only σ-continuous but also τ-continuous functionals.

The purpose of the study and the benefits of the results

The purpose of the study is to

- find a representing measure that is inner σ-continuous and outer σ-continuous on the respective collections of tractable sets such as the open, the closed, the compact, and the measurable sets. Moreover, in addition to the σ-continuity, discuss the τ-continuity of the representing measures.
- formulate continuous Choquet integral representation theorems in an abstract setting so as to contain as many existing results as possible.
- clarify the relation between regularity and continuity of nonadditive measures.

The benefits of the results are as follows:

- the representing measures are simultaneously inner and outer continuous.
- the respective collections of sets for which the representing measures are inner and outer continuous are larger than those in previous studies.
- the regularity of the representing measures is also clarified.
- it is possible to handle not only σ-continuous but also τ-continuous functionals.

Some characteristics of functional

We introduce some characteristics of a functional that both of Ch and Ch^a have.

Definition 5

Let $I\colon \Phi \to [0,\infty]$ for $\Phi \subset [0,\infty]^X$ or $\Phi \subset \mathbb{R}^X$. We say that I is

- monotone if $I(f) \leq I(g)$ whenever $f, g \in \Phi$ and $f \leq g$.
- positively homogeneous if $I(cf) = cI(f)$ whenever $f \in \Phi$, $c > 0$. and $cf \in \Phi$.
- bounded if there is $M > 0$ such that $|I(f)| \leq M\|f\|_\infty$ for all $f \in \Phi$.
- translatable if $I(f + c) = I(f) + I(c)$ whenever $f \in \Phi$, $c \in \mathbb{R}$, and $c, f + c \in \Phi$.
- comonotonically additive if $I(f + g) = I(f) + I(g)$ whenever $f, g \in \Phi$, $f + g \in \Phi$, and f and g are comonotonic[a], i.e.,

$$\forall x_1, x_2 \in X,\ f(x_1) < f(x_2) \Rightarrow g(x_1) \leq g(x_2).$$

[a]It is called similarly ordered in “Inequalities” written by Hardy, Littlewood, and Pólya.

The comonotonic additivity of a functional is very important and has been used to formulate the Choquet integral representation theorems.

The following asymptotic translatability of a functional will be needed to obtain the asymmetric Choquet integral representation theorem for functionals defined on the function spaces without nonzero constant functions such as the spaces $C_{00}(X)$ and $C_0(X)$ for a locally compact X.

Definition 6

Let $I \colon \Phi \to [0, \infty]$ for $\Phi \subset [0, \infty]^X$ or $\Phi \subset \mathbb{R}^X$. We say that I is

- asymptotically τ-translatable if

$$\lim_{\gamma \in \Gamma} I(f + g_\gamma) = I(f) + \lim_{\gamma \in \Gamma} I(g_\gamma)$$

  whenever $f \in \Phi$, $\{g_\gamma\}_{\gamma \in \Gamma}$ is a nondecreasing net in Φ^+ such that $f + g_\gamma \in \Phi^+$ for all $\gamma \in \Gamma$ and $\{g_\gamma\}_{\gamma \in \Gamma}$ converges pointwise to a constant.
- asymptotically σ-translatable if the same holds for the case $\{g_\gamma\}_{\gamma \in \Gamma}$ being a sequence.

The asymptotic τ-translatability holds for $\mathrm{Ch}^a(\mu, \cdot)$ if μ is inner τ-continuous on the collection $\{\{f > t\} \colon f \in \Phi^+, t > 0\}$.

The main tools in this study are our improvement of the Greco theorem[1] on the Choquet integral representation theorem in a general setting and König's scheme[2] on a "continuous Daniell-Stone representation theorem".

We first introduce the Greco theorem that gives a primitive form of the Choquet integral representation theorem.

Let X be a nonempty set and $\Phi \subset [0,\infty]^X$ with $0 \in \Phi$. Given a monotone functional $I\colon \Phi \to [0,\infty]$ with $I(0) = 0$, define the set functions $\alpha, \beta\colon 2^X \to [0,\infty]$ by

$$\alpha(A) := \sup\{I(f)\colon f \in \Phi, f \le \chi_A\},$$

$$\beta(A) := \inf\{I(f)\colon f \in \Phi, \chi_A \le f\}$$

for every $A \subset X$. Then, they are nonadditive measures on 2^X with $\alpha \le \beta$.

[1] G.H. Greco, Sulla rappresentazione di funzionali mediante integrali, Rend. Sem. Mat. Univ. Padova 66 (1982) 21–42.

[2] H. König, Measure and Integration, An Advanced Course in Basic Procedures and Applications, Chapter V, Springer 1997.

Theorem 7 (Greco 1982)

Let $\mathcal{D}$ be a collection of subsets of X with $\emptyset \in \mathcal{D}$. Let $\Phi \subset [0,\infty]^X$ be positively homogeneous and Stonean, i.e.,

$$cf, f \wedge c, f - f \wedge c = (f-c)^+ \in \Phi$$

for any $f \in \Phi$ and $c > 0$. If a functional $I\colon \Phi \to [0,\infty]$ with $I(0) = 0$ satisfies

1. (i) $I(f) \leq I(g)$ whenever $f, g \in \Phi$ and $f \leq g$ (monotone),
2. (ii) $I(f) = I(f \wedge c) + I(f - f \wedge c)$ for any $f \in \Phi$ and $c > 0$ (horizontally additive),
3. (iii) $I(f) = \sup_{a>0} I(f - f \wedge a) = \sup_{b>0} I(f \wedge b)$ for any $f \in \Phi$ (marginal continuous),

then, for every nonadditive measure λ on $\mathcal{D}$ the following are equivalent.

- (a) $\alpha(D) \leq \lambda(D) \leq \beta(D)$ for all $D \in \mathcal{D}$.
- (b) $I(f) = \mathrm{Ch}(\lambda, f)$ for all $f \in \Phi$.

The horizontal additivity corresponds to the fact that the Lebesgue integral can be split in the middle of an integral interval, and the marginal continuity corresponds to the fact that the Lebesgue integral is continuous at lower and upper endpoints of the integral.

König's scheme (1997)

Recall König's scheme on a continuous Daniell-Stone representation appeared in his book in 1997.

Let X be a nonempty set. For $\Phi \subset [0,\infty]^X$, define the following collections of sets.

$$\mathcal{H}_\Phi := \Big\{\{f > t\} \colon f \in \Phi, t > 0\Big\}$$

$$\mathcal{H}_\Phi^\sigma := \left\{\bigcup_{H \in \mathcal{H}_0} H \colon \mathcal{H}_0 \text{ is an at most countable subcollection of } \mathcal{H}_\Phi\right\}$$

$$\mathcal{H}_\Phi^\tau := \left\{\bigcup_{H \in \mathcal{H}_0} H \colon \mathcal{H}_0 \text{ is an arbitrary subcollection of } \mathcal{H}_\Phi\right\}$$

$$\mathcal{L}_\Phi := \Big\{\{f \geq t\} \colon f \in \Phi, t > 0\Big\}$$

$$\mathcal{L}_\Phi^\sigma := \left\{\bigcap_{L \in \mathcal{L}_0} L \colon \mathcal{L}_0 \text{ is an at most countable subcollection of } \mathcal{L}_\Phi\right\}$$

$$\mathcal{L}_\Phi^\tau := \left\{\bigcap_{L \in \mathcal{L}_0} L \colon \mathcal{L}_0 \text{ is an arbitrary subcollection of } \mathcal{L}_\Phi\right\}$$

Recall that

$$\alpha(D) := \sup\{I(f) \colon f \in \Phi,\ f \le 1_D\},$$
$$\beta(D) := \inf\{I(f) \colon f \in \Phi,\ 1_D \le f\}.$$

Then, for $\bullet = \sigma\tau$, using the collections $\mathcal{H}_\Phi^\bullet$ and $\mathcal{L}_\Phi^\bullet$ of sets, define the regularizations of α and β by

$$\alpha^\bullet(D) := \inf\left\{\alpha(H) \colon D \subset H,\ H \in \mathcal{H}_\Phi^\bullet\right\},$$
$$\beta^\bullet(D) := \sup\left\{\beta(L) \colon L \subset D,\ L \in \mathcal{L}_\Phi^\bullet\right\}$$

for every $D \subset X$, which are also nonadditive measures on 2^X.

The combination of the Greco theorem and the König scheme leads to

Proposition 8 (K: 2024)

Let $\Phi \subset [0,\infty]^X$ with $0 \in \Phi$. Assume that

i. Φ is positively homogeneous and Stonean, i.e., $cf, f \wedge c, f - f \wedge c \in \Phi$ for any $f \in \Phi$ and $c > 0$.
ii. Φ separates sets in $\mathcal{L}_\Phi^\bullet$ and $\mathcal{H}_\Phi^\bullet$, i.e., for any $L \in \mathcal{L}_\Phi^\bullet$ and $H \in \mathcal{H}_\Phi^\bullet$ with $L \subset H$, there is an $f \in \Phi$ such that $1_L \leq f \leq 1_H$.

Let a monotone functional $I \colon \Phi \to [0,\infty]$ be horizontally additive and marginal continuous with $I(0) = 0$. Then the following hold.

1. If $\bullet = \sigma$, then
$$\alpha \leq \beta^\sigma \leq \alpha^\sigma \leq \beta,$$
and if $\bullet = \tau$, then
$$\alpha \leq \beta^\sigma \leq \beta^\tau \leq \alpha^\tau \leq \alpha^\sigma \leq \beta,$$
so they are all representing measures of I.

Proposition 8 (Continued)

2. $\alpha^\bullet(L) = \beta(L)$ for every $L \in \mathcal{L}_\Phi^\bullet$.
3. $\alpha(H) = \beta^\bullet(H)$ for every $H \in \mathcal{H}_\Phi^\bullet$.
4. $\alpha^\bullet$ is inner $\mathcal{L}_\Phi^\bullet$ regular on $\mathcal{H}_\Phi^\bullet$, i.e., for any $H \in \mathcal{H}_\Phi^\bullet$ we have
$$\alpha^\bullet(H) = \sup\left\{\alpha^\bullet(L) \colon L \subset H, L \in \mathcal{L}_\Phi^\bullet\right\}.$$
5. $\beta^\bullet$ is outer $\mathcal{H}_\Phi^\bullet$ regular on $\mathcal{L}_\Phi^\bullet$, i.e., for any $L \in \mathcal{L}_\Phi^\bullet$ we have
$$\beta^\bullet(L) = \inf\left\{\beta^\bullet(H) \colon L \subset H, H \in \mathcal{H}_\Phi^\bullet\right\}.$$

To obtain the continuous Choquet integral representation theorem, which is our main topic in this talk, we need to prepare two more results:

- The relationship between regularity and continuity of nonadditive measures.
- The continuity of nonadditive measures α and β defined by the functional I.

Definition 9

A collection $\mathcal{K}$ of subsets of X is called τ-compact if every subcollection of $\mathcal{K}$ whose intersection is empty has a further finite subcollection whose intersection is empty. It is called σ-compact if the same holds for any at most countable subcollection of $\mathcal{K}$.

Definition 10

A functional $I\colon \Phi \to [0,\infty]$ is called

- inner τ-continuous if $I(f_\gamma) \to I(f)$ whenever $\{f_\gamma\}_{\gamma\in\Gamma}$ is a net in Φ, $f \in \Phi$, and $f_\gamma \uparrow f$.
- outer τ-continuous if $I(f_\gamma) \to I(f)$ whenever $\{f_\gamma\}_{\gamma\in\Gamma}$ is a net in Φ, $f \in \Phi$, and $f_\gamma \downarrow f$.
- conditionally outer τ-continuous if $I(f_\gamma) \to I(f)$ whenever $\{f_\gamma\}_{\gamma\in\Gamma}$ is a net in Φ, $f \in \Phi$, $f_\gamma \downarrow f$, and $I(f_{\gamma_0}) < \infty$ for some $\gamma_0 \in \Gamma$.
- The inner (resp. outer, conditionally outer) σ-continuity is defined with "net" replaced by "sequence".

Proposition 11 (K.: 2024)

Let $\Phi \subset [0,\infty]^X$ be positively homogeneous and Stonean with $0 \in \Phi$. Let μ be a nonadditive measure on 2^X.

(1) Assume that Φ is a lattice.

1. μ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet \Rightarrow \mu$ is inner $\mathcal{L}_\Phi^\bullet$ regular on $\mathcal{H}_\Phi^\bullet$.
2. μ is outer $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet \Rightarrow \mu$ is outer $\mathcal{H}_\Phi^\bullet$ regular on $\mathcal{L}_\Phi^\bullet$.

(2) Assume that $(1-f)\wedge g \in \Phi$ whenever $f, g \in \Phi$ and $0 \leq f \leq 1$ and that $\mathcal{L}_\Phi$ is $\bullet$-compact.

1. μ is inner $\mathcal{L}_\Phi^\bullet$ regular on $\mathcal{H}_\Phi^\bullet \Rightarrow \mu$ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$.
2. μ is outer $\mathcal{H}_\Phi^\bullet$ regular on $\mathcal{L}_\Phi^\bullet \Rightarrow \mu$ is outer $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$.

The improved Greco theorem (Prop. 8) and Prop. 11 show that if $\mathcal{L}_\Phi$ is $\bullet$-compact, then $\alpha^\bullet$ is a representing measure we are looking for.

Indeed, by (2) of Prop. 11

- $\alpha^\bullet$ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$ since it is inner $\mathcal{L}_\Phi^\bullet$ regular on $\mathcal{H}_\Phi^\bullet$ by ④ of Prop. 8,
- $\beta^\bullet$ is outer $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$ since it is outer $\mathcal{H}_\Phi^\bullet$ regular on $\mathcal{L}_\Phi^\bullet$ by ⑤ of Prop. 8,
- $\alpha^\bullet = \beta^\bullet$ on $\mathcal{L}_\Phi^\bullet$ by ② of Prop. 8 since $\beta(L) = \beta^\bullet(L)$ for any $L \in \mathcal{L}_\Phi^\bullet$

Therefore, $\alpha^\bullet$ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$ and outer $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$.

The continuity of α and β follows from the continuity of I

Proposition 12 (K: 2024)

Let $\Phi \subset [0,\infty]^X$ be a positively homogeneous Stonean lattice with $0 \in \Phi$. Let $I\colon \Phi \to [0,\infty]$ be a monotone functional.

1. I is inner $\bullet$-continuous on $\Phi \Rightarrow \alpha$ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$.
2. I is outer (resp. conditionally outer) $\bullet$-continuous on $\Phi \Rightarrow \beta$ is outer (resp. conditionally outer) $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$.

The improved Greco theorem (Prop. 8) and Prop. 12 show that if I is inner and outer $\bullet$-continuous on Φ, then $\alpha^\bullet$ is a representing measure we are looking for.

Indeed, by Prop. 12

- $\alpha^\bullet$ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$ since I is inner $\bullet$-continuous on Φ,
- $\beta^\bullet$ is outer (resp. conditionally outer) $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$ since I is outer (resp. conditionally outer) $\bullet$-continuous on Φ,
- $\alpha^\bullet = \beta^\bullet$ on $\mathcal{L}_\Phi^\bullet$ by ② of Prop. 8 since $\beta(L) = \beta^\bullet(L)$ for any $L \in \mathcal{L}_\Phi^\bullet$.

Therefore, $\alpha^\bullet$ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$ and outer $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$.

Two types of abstract forms of the representation theorems

In the following, we assume that

- $\Phi \subset [0,\infty]^X$ is positively homogeneous Stonean with $0 \in \Phi$.
- $I\colon \Phi \to [0,\infty]$ is a horizontally additive, marginal continuous monotone functional with $I(0) = 0$.

Theorem 13 (K. 2024) for the case where X is endowed with a topology

Let Φ separate sets in $\mathcal{L}_\Phi^\bullet$ and $\mathcal{H}_\Phi^\bullet$. Assume that either of ① or ② below.

1. Φ is a lattice and I is $\bullet$-continuous on Φ (continuity condition).
2. Φ satisfies the following conditions (compactness condition).
   - (i) $(1-f)\wedge g \in \Phi$ for any $f,g \in \Phi$ with $0 \le f \le 1$.
   - (ii) $\mathcal{L}_\Phi$ is $\bullet$-compact.

Then, there is a unique nonadditive measure $\mu\colon 2^X \to [0,\infty]$ such that

- (a) $I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in \Phi$.
- (b) μ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$ and outer $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$.
- (c) μ is inner $\mathcal{L}_\Phi^\bullet$ regular on $\mathcal{H}_\Phi^\bullet$ and outer $\mathcal{H}_\Phi^\bullet$ regular on 2^X.
- (d) If I is bounded and Φ possesses an approximate $\bullet$-identity, then μ is finite.

Unlike the previous theorem, the following theorem yields a representing measure that is inner and outer $\bullet$-continuous on the same collection $\mathcal{D}$ of subsets of X.

Theorem 14 (K. 2024) for the case where X is not endowed with a topology

Let $\mathcal{D}$ be a collection of subsets of X with $\emptyset \in \mathcal{D}$. Assume that

- (i) $1_D \in \Phi$ for every $D \in \mathcal{D}$.
- (ii) I is $\bullet$-continuous on Φ.

Then, there is a unique nonadditive measure $\mu \colon 2^X \to [0, \infty]$ such that

- (a) $I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in \Phi$.
- (b) μ is inner and outer $\bullet$-continuous on $\mathcal{D}$.
- (c) μ is outer $\mathcal{H}_\Phi^\bullet$ regular on 2^X.
- (d) μ is finite if and only if I is bounded.

A variety of continuous Choquet integral representation theorems

We introduce various continuous Choquet integral representation theorems that can be derived from our abstract forms of the representation theorems.

In what follows, $\mathcal{G}$, $\mathcal{F}$, and $\mathcal{K}$ are the collections of the open sets, the closed sets, and the compact sets, respectively.

1. X is compact and $I\colon C_b(X) \to \mathbb{R}$ is C.M. $\Rightarrow \mu$ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$: an improvement of Zhou [1].

When X is a normal space, we should assume the continuity of a functional for the representing measure to be continuous.

2. X is normal and $I\colon C_b(X) \to \mathbb{R}$ is C.M. and τ-continuous on $C_b^+(X)$ $\Rightarrow \mu$ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{F}$: a result of K [2].

[1] L. Zhou, Trans. Amer. Math. Soc. 350 (1998) 1811-1822.
[2] J. Kawabe, under review

When X is locally compact, we have the following four results.

3. $I\colon C_{00}^{+}(X) \to [0,\infty)$ is C.M. $\Rightarrow \mu$ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$: an improvement of Anger [3] and Narukawa [4].
4. $I\colon C_{0}^{+}(X) \to [0,\infty)$ is bounded C.M. $\Rightarrow \mu$ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$: an improvement of K [5].

To obtain a representing measure of the functional I on $C_{00}(X)$ and $C_0(X)$, we need to assume the asymptotic translatability of I since, unlike $C_b(X)$, the spaces $C_{00}(X)$ and $C_0(X)$ do not contain nonzero constant functions.

5. $I\colon C_{00}(X) \to \mathbb{R}$ is asymptotically τ-translatable, bounded C.M. $\Rightarrow \mu$ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$: an improvement of K [5].
6. $I\colon C_{0}(X) \to \mathbb{R}$ is asymptotically τ-translatable, bounded C.M. $\Rightarrow \mu$ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$: an improvement of K [6].

[3] B. Anger, Math. Ann. 229 (1977) 245-258.
[4] Y. Narukawa, Fuzzy Sets and Systems 158 (2007) 963-972.
[5] J. Kawabe, Nonlinear Mathematics for Uncertainty and its Applications, 2011, pp.35-42.
[6] J. Kawabe, J. Approx. Reason. 54 (2013) 418-426.

Next we consider the case where X is not endowed with a topology.

7. $\mathcal{D}$ is a collection of subsets of X with $\emptyset, X \in \mathcal{D}$, $B(X, \mathcal{D})$ is the collection of all bounded functions $f\colon X \to \mathbb{R}$ such that $\{f > t\} \in \mathcal{D}$ for every $t \in \mathbb{R}$, and $I\colon B(X, \mathcal{D}) \to \mathbb{R}$ is C.M. and τ-continuous on $B^+(X, \mathcal{D})$ $\Rightarrow \mu$ is inner and outer τ-continuous on $\mathcal{D}$: a generalization of Schmeidler [7] and a generalization of the monotone versions of Murofushi et al. [8] and Rébillé [9].

Finally we also obtain

8. the corresponding generalizations of the abstract forms of the Choquet integral representation theorems of Šipoš [10], Zhou [1], König [11], and a monotone version of Vioglio et al. [12],
9. the σ-continuous versions of all of the above results.

[7] D. Schmeidler, Proc. Amer. Math. Soc. 97 (1986) 255-261.
[8] T. Murofushi, M. Sugeno, M. Machida, Fuzzy Sets and Systems 64 (1994) 73-86.
[9] Y. Rébillé, Fuzzy Sets and Systems 153 (2005) 79-94.
[10] J. Šipoš, Math. Slovaca 29 (1979) 333-345.
[11] H. König, Ann. Univ. Sarav. Ser. Math. 9 (1998) 123-153.
[12] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, L. Montrucchio, J. Math. Anal. Appl. 385 (2012) 895-912.

References I

- B. Anger, Representation of capacities, Math. Ann. 229 (1977) 245–258.
- G. Choquet, Theory of capacities. Ann. Inst. Fourier (Grenoble) 5 (1953–1954) 131–295.
- C. Dellacherie, Quelques commentaires sur les prolongements de capaciés, Séminaire de Probabiliés V, Strasbourg, Lecture Notes in Math. 191, Springer, Berlin, 77–81, 1970.
- G.H. Greco, Sulla rappresentazione di funzionali mediante integrali, Rend. Semin. Mat. Univ. Padova 66 (1982) 21–42.
- J. Kawabe, Riesz type integral representations for comonotonically additive functionals, in: L. Shoumei, X. Wang, Y. Okazaki, J. Kawabe, T. Murofushi, L. Guan (Eds.), Nonlinear Mathematics for Uncertainty and its Applications, Springer, Berlin, 2011, pp. 35–42.
- J. Kawabe, The Choquet integral representability of comonotonically additive functionals in locally compact spaces, Internat. J. Approx. Reason. 54 (2013) 418–426.
- H. König, Measure and Integration, An Advanced Course in Basic Procedures and Applications, Springer, Berlin, 1997.
- H. König, Measure and integration; integral representations of isotone functionals, Ann. Univ. Sarav. Ser. Math. 9 (1998) 123–153.
- H. König, Measure and integration: an attempt at unified systematization, Rend. Istit. Mat. Univ. Trieste 34 (2002) 155–214.
- T. Murofushi, M. Sugeno, M. Machida, Non-monotonic fuzzy measures and the Choquet integral, Fuzzy Sets and Systems 64 (1994) 73–86.

References II

- Y. Narukawa, T. Murofushi, M. Sugeno, Regular fuzzy measure and representation of comonotonically additive functional, Fuzzy Sets and Systems 112 (2000) 177–186.
- Y. Narukawa, T. Murofushi, Conditions for Choquet integral representation of the comonotonically additive and monotone functional, J. Math. Anal. Appl. 282 (2003) 201–211.
- Y. Narukawa, Inner and outer representation by Choquet integral, Fuzzy Sets and Systems 158 (2007) 963–972.
- Y. Rébillé, Sequentially continuous non-monotonic choquet integrals, Fuzzy Sets and Systems 153 (2005) 79–94.
- D. Schmeidler, Integral representation without additivity, Proc. Amer. Math. Soc. 97 (1986) 255–261.
- J. Šipoš, Integral representations of non-linear functionals, Math. Slovaca 29 (1979) 333–345.
- M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. Dissertation, Tokyo Inst. of Tech., Tokyo, 1974.
- M. Sugeno, Y. Narukawa, T. Murofushi, Choquet integral and fuzzy measures on locally compact space, Fuzzy Sets and Systems 99 (1998) 205–211.
- S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, L. Montrucchio, Signed integral representations of comonotonic additive functionals, J. Math. Anal. Appl. 385 (2012) 895–912.
- L. Zhou, Integral representation of continuous comonotonically additive functionals, Trans. Amer. Math. Soc. 350 (1998) 1811-1822.

Thank you so much for your kind attention!

How to derive a result of Zhou (1998) for $I\colon C_b(X) \to \mathbb{R}$ and X: compact.

- X is a compact Hausdorff space.
- $C_b(X)$ is the vector lattice of all (bounded) continuous functions $f\colon X \to \mathbb{R}$.
- $\Phi := C_b^+(X)$.

Then

- $\mathcal{H}_\Phi = \mathcal{H}_\Phi^\sigma$ and they are the collection of the open F_σ sets,
- $\mathcal{H}_\Phi^\tau$ is the collection $\mathcal{G}$ of the open sets,
- $\mathcal{L}_\Phi = \mathcal{L}_\Phi^\sigma$ and they are the collection of the compact G_δ sets,
- $\mathcal{L}_\Phi^\tau$ is the collection $\mathcal{K}$ of the compact sets,

and

- Φ is positively homogeneous and Stonean with $0 \in \Phi$,
- $1 \in \Phi$, so Φ has an approximate τ-identity,
- Any C.M. real-valued functional I on Φ is horizontally additive, marginal continuous, and bounded with $I(0) = 0$,
- Φ separates sets in $\mathcal{K}$ and $\mathcal{G}$ by the Urysohn theorem,
- $(1 - f) \wedge g \in \Phi$ whenever $f, g \in \Phi$ and $0 \leq f \leq 1$,
- $\mathcal{L}_\Phi^\tau = \mathcal{K}$ is compact.

By Theorem 19 (compact case) we first obtain an intermediate result.

Theorem 13 (K: 2024)

Let X be a compact Hausdorff space. Let $I\colon C_b^+(X) \to [0,\infty)$ be a C.M. functional. Then there is a unique finite nonadditive measure μ on 2^X such that

- (a) $I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in C_b^+(X)$,
- (b) μ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$,
- (c) μ is inner $\mathcal{K}$ regular on $\mathcal{G}$ and outer $\mathcal{G}$ regular on 2^X.

For any $f \in C_b(X)$ there is a $c > 0$ such that $f + c \in C_b^+(X)$. Hence, by ⓐ of Theorem 13 and the translatability of Ch^a, we have

$$I(f + c) = \mathrm{Ch}(\mu, f + c) = \mathrm{Ch}^a(\mu, f) + \mathrm{Ch}(\mu, c) = \mathrm{Ch}^a(\mu, f) + I(c). \qquad (3)$$

Since any C.M. functional is translatable, that is,

$$I(f + c) = I(f) + I(c), \qquad (4)$$

it follows from (3) and (4) that $I(f) = \mathrm{Ch}^a(\mu, f)$.

Hence, we obtain an improvement of Zhou (1998).

Theorem 14 (K: 2024)

Let X be a compact Hausdorff space. Let $I\colon C_b(X) \to \mathbb{R}$ be a C.M. functional. Then there is a unique finite nonadditive measure μ on 2^X such that

- (a) $I(f) = \mathrm{Ch}^a(\mu, f)$ for every $f \in C_b(X)$,
- (b) μ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$,
- (c) μ is inner $\mathcal{K}$ regular on $\mathcal{G}$ and outer $\mathcal{G}$ regular on 2^X.

- X is a locally compact Hausdorff space.
- $C_{00}(X)$ is the vector lattice of all continuous $f\colon X \to \mathbb{R}$ with compact support.
- $\Phi := C_{00}^{+}(X)$.

Then

- $\mathcal{H}_\Phi$ is the collection of the bounded open K_σ sets,
- $\mathcal{H}_\Phi^\sigma$ is the collection of the open K_σ sets,
- $\mathcal{H}_\Phi^\tau$ is the collection $\mathcal{G}$ of the open sets,
- $\mathcal{L}_\Phi = \mathcal{L}_\Phi^\sigma$ and they are the compact G_δ sets,
- $\mathcal{L}_\Phi^\tau$ is the collection $\mathcal{K}$ of the compact sets,

and

- Φ is positively homogeneous and Stonean with $0 \in \Phi$,
- Φ has an approximate τ-identity,
- Any C.M. real-valued functional I on Φ is horizontally additive and marginal continuous with $I(0) = 0$,
- Φ separates sets in $\mathcal{K}$ and $\mathcal{G}$ by the Urysohn theorem,
- $(1 - f) \wedge g \in \Phi$ whenever $f, g \in \Phi$ and $0 \le f \le 1$,
- $\mathcal{L}_\Phi^\tau = \mathcal{K}$ is compact.

Hence, we obtain an improvement of Narukawa (2007) by Theorem 19 (compact case).

Theorem 15 (K: 2011)

Let X be a locally compact Hausdorff space. Let $I\colon C_{00}^{+}(X) \to \mathbb{R}$ be a C.M. functional. Then there is a unique nonadditive measure μ on 2^X such that

- ⓐ $I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in C_{00}^{+}(X)$,
- ⓑ μ is τ-inner continuous on $\mathcal{G}$ and τ-outer continuous on $\mathcal{K}$.
- ⓒ μ is inner $\mathcal{K}$ regular on $\mathcal{G}$ and outer $\mathcal{G}$ regular on 2^X.

Since $C_{00}(X)$ has a τ-approximate identity, for any $f \in C_{00}(X)$, one can find a net $\{g_\gamma\}_{\gamma\in\Gamma}$ in $C_{00}^{+}(X)$ and a constant $c > 0$ such that $f + g_\gamma \in C_{00}^{+}(X)$ for all $\gamma \in \Gamma$ and $g_\gamma \uparrow c$. Then, by ⓐ and ⓑ of Theorem 15 we have

$$\begin{aligned}\lim_{\gamma\in\Gamma} I(f+g_\gamma) &= \lim_{\gamma\in\Gamma} \mathrm{Ch}(\mu, f+g_\gamma) = \mathrm{Ch}(\mu, f+c) = \mathrm{Ch}^a(\mu, f) + \mathrm{Ch}(\mu, c)\\ &= \mathrm{Ch}^a(\mu, f) + \lim_{\gamma\in\Gamma} \mathrm{Ch}(\mu, g_\gamma) = \mathrm{Ch}^a(\mu, f) + \lim_{\gamma\in\Gamma} I(g_\gamma). \end{aligned} \tag{5}$$

If I is asymptotically τ-translatable, then

$$\lim_{\gamma\in\Gamma} I(f+g_\gamma) = I(f) + \lim_{\gamma\in\Gamma} I(g_\gamma), \tag{6}$$

so that it follows from (5) and (6) that

$$I(f) = \mathrm{Ch}^a(\mu, f).$$

Moreover, if I is bounded, then μ is finite. Hence, we obtain an improvement of K. (2013) by Theorem 15.

Theorem 16 (K: 2013)

Let X be a locally compact space. Let $I\colon C_{00}(X) \to \mathbb{R}$ be an asymptotically τ-translatable, bounded- C.M. functional. Then there exists a unique finite nonadditive measure μ on 2^X such that

- a $I(f) = \mathrm{Ch}^a(\mu, f)$ for every $f \in C_{00}(X)$,
- b μ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$,
- c μ is inner $\mathcal{K}$ regular on $\mathcal{G}$ and outer $\mathcal{G}$ regular on 2^X.

- X is a locally compact Hausdorff space.
- $C_0(X)$ is the vector lattice of all continuous $f\colon X \to \mathbb{R}$ vanishing at infinity.
- $\Phi := C_0^+(X)$.

Then

- $\mathcal{H}_\Phi$ is the collection of the bounded open K_σ sets,
- $\mathcal{H}_\Phi^\sigma$ is the collection of the open K_σ sets,
- $\mathcal{H}_\Phi^\tau$ is the collection $\mathcal{G}$ of the open sets,
- $\mathcal{L}_\Phi = \mathcal{L}_\Phi^\sigma$ and they are the compact G_δ sets,
- $\mathcal{L}_\Phi^\tau$ is the collection $\mathcal{K}$ of the compact sets,

and

- Φ is positively homogeneous and Stonean with $0 \in \Phi$,
- Φ has an approximate τ-identity,
- Any C.M. real-valued functional I on Φ is horizontally additive and marginal continuous with $I(0) = 0$,
- Φ separates sets in $\mathcal{K}$ and $\mathcal{G}$ by the Urysohn theorem,
- $(1 - f) \wedge g \in \Phi$ whenever $f, g \in \Phi$ and $0 \leq f \leq 1$,
- $\mathcal{L}_\Phi^\tau = \mathcal{K}$ is compact.

By Theorem 19 (compact case) we obtain an intermediate result.

Theorem 17 (K: 2013)

Let X be a locally compact Hausdorff space. Let $I\colon C_0^+(X) \to \mathbb{R}$ be a bounded C.M. functional. Then there is a unique nonadditive measure μ on 2^X such that

- ⓐ $I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in C_0^+(X)$,
- ⓑ μ is τ-inner continuous on $\mathcal{G}$ and τ-outer continuous on $\mathcal{K}$.
- ⓒ μ is inner $\mathcal{K}$ regular on $\mathcal{G}$ and outer $\mathcal{G}$ regular on 2^X.

Since $C_0(X)$ has a τ-approximate identity, for any $f \in C_0(X)$, one can find a net $\{g_\gamma\}_{\gamma\in\Gamma}$ in $C_0^+(X)$ and a constant $c > 0$ such that $f + g_\gamma \in C_0^+(X)$ for all $\gamma \in \Gamma$ and $g_\gamma \uparrow c$ Then, by ⓐ and ⓑ of Theorem 17 we have

$$\begin{aligned}\lim_{\gamma\in\Gamma} I(f+g_\gamma) &= \lim_{\gamma\in\Gamma} \mathrm{Ch}(\mu, f+g_\gamma) = \mathrm{Ch}(\mu, f+c) = \mathrm{Ch}^a(\mu, f) + \mathrm{Ch}(\mu, c) \\ &= \mathrm{Ch}^a(\mu, f) + \lim_{\gamma\in\Gamma} \mathrm{Ch}(\mu, g_\gamma) = \mathrm{Ch}^a(\mu, f) + \lim_{\gamma\in\Gamma} I(g_\gamma). \qquad (7)\end{aligned}$$

If I is asymptotically τ-translatable, then

$$\lim_{\gamma\in\Gamma} I(f+g_\gamma) = I(f) + \lim_{\gamma\in\Gamma} I(g_\gamma), \tag{8}$$

so that it follows from (7) and (8) that

$$I(f) = \mathrm{Ch}^a(\mu, f).$$

Moreover, if I is bounded, then μ is finite. Hence, we obtain an improvement of K. (2013) by Theorem 12.

Theorem 18 (K: 2013)

Let X be a locally compact space. Let $I\colon C_0(X) \to \mathbb{R}$ be an asymptotically τ-translatable, bounded C.M. functional. Then there is a unique finite nonadditive measure μ on 2^X such that

- (a) $I(f) = \mathrm{Ch}^a(\mu, f)$ for every $f \in C_0(X)$,
- (b) μ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{K}$.
- (c) μ is inner $\mathcal{K}$ regular on $\mathcal{G}$ and outer $\mathcal{G}$ regular on 2^X.

How to derive a result of K (2024) for $I\colon C_b(X) \to \mathbb{R}$ and X: normal.

- X is a normal space.
- $C_b(X)$ is the vector lattice of all bounded continuous functions $f\colon X \to \mathbb{R}$.
- $\Phi := C_b^+(X)$.

Then

- $\mathcal{H}_\Phi = \mathcal{H}_\Phi^\sigma$ and they are the collection of the open F_σ sets.
- $\mathcal{H}_\Phi^\tau$ is the collection $\mathcal{G}$ of the open sets.
- $\mathcal{L}_\Phi = \mathcal{L}_\Phi^\sigma$ and they are the collection of the closed G_δ sets.
- $\mathcal{L}_\Phi^\tau$ is the collection $\mathcal{F}$ of the closed sets.

and

- $1 \in \Phi$, so Φ has an approximate τ-identity.
- Any τ-continuous C.M. real-valued functional I on Φ is horizontally additive, marginal continuous, and bounded with $I(0) = 0$.
- Φ separates sets in $\mathcal{F}$ and $\mathcal{G}$ by the Urysohn theorem.

By Theorem 19 (cotinuous case) we first obtain an intermediate result.

Theorem 19 (K: 2024)

Let X be a normal space. Let $I\colon C_b^+(X) \to [0,\infty)$ be a τ-continuous C.M. functional. Then there is a unique finite nonadditive measure μ on 2^X such that

- (a) $I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in C_b^+(X)$,
- (b) μ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{F}$,
- (c) μ is inner $\mathcal{F}$ regular on $\mathcal{G}$ and outer $\mathcal{G}$ regular on 2^X.

For any $f \in C_b(X)$ there is a $c > 0$ such that $f + c \in C_b^+(X)$. Hence, by (a) of Theorem 19 and the translatability of Ch^a, we have

$$I(f+c) = \mathrm{Ch}(\mu, f+c) = \mathrm{Ch}^a(\mu, f) + \mathrm{Ch}(\mu, c) = \mathrm{Ch}^a(\mu, f) + I(c). \tag{9}$$

Since any C.M. functional is translatable, that is,

$$I(f+c) = I(f) + I(c), \tag{10}$$

it follows from (9) and (10) that $I(f) = \mathrm{Ch}^a(\mu, f)$.

Hence we obtain a new result of K. (2024).

Theorem 20 (K: 2024)

Let X be a normal space. Let $I\colon C_b(X) \to \mathbb{R}$ be a τ-continuous C.M. functional. Then there is a unique finite nonadditive measure μ on 2^X such that

a. $I(f) = \mathrm{Ch}^a(\mu, f)$ for every $f \in C_b(X)$,
b. μ is inner τ-continuous on $\mathcal{G}$ and outer τ-continuous on $\mathcal{F}$,
c. μ is inner $\mathcal{F}$ regular on $\mathcal{G}$ and outer $\mathcal{G}$ regular on 2^X.

- X is any set.
- $\mathcal{D}$ is a collection of subsets of X with $\emptyset \in \mathcal{D}$.
- $B(X, \mathcal{D})$ is the space of all bounded functions $f \colon X \to \mathbb{R}$ such that $\{f > t\} \in \mathcal{D}$ for every $t \in \mathbb{R}$.
- $\Phi := B^+(X, \mathcal{D})$.

Then

- Φ is positively homogeneous and Stonean with $0 \in \Phi$,
- $\mathcal{H}_\Phi = \mathcal{L}_\Phi = \mathcal{D}$,
- $\mathcal{H}_\Phi^\sigma = \mathcal{L}_\Phi^\sigma = \mathcal{D}$ if $\mathcal{D}$ is a σ-algebra,

and

- $1_D \in \Phi$ for every $D \in \mathcal{D}$,
- Any C.M. real-valued functional I on Φ is horizontally additive and marginal continuous with $I(0) = 0$.

By Theorem 20 we first obtain an intermediate result.

Theorem 21 (K: 2024)

Let X be a nonempty set and $\mathcal{D}$ a collection of subsets of X with $\emptyset \in \mathcal{D}$. Let $I\colon B^+(X,\mathcal{D}) \to [0,\infty)$ be a τ-continuous C.M. functional. Then there is a unique nonadditive measure μ on 2^X such that

- (a) $I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in B^+(X,\mathcal{D})$,
- (b) μ is inner and outer τ-continuous on $\mathcal{D}$,
- (c) μ is outer $\mathcal{D}$ regular on 2^X.

Assume that $X \in \mathcal{D}$. Since $1 \in B(X,\mathcal{D})$, for any $f \in B(X,\mathcal{D})$ there is a $c > 0$ such that $f + c \in B^+(X,\mathcal{D})$. Hence, by (a) of Theorem 21 and the translatability of Ch^a,

$$I(f+c) = \mathrm{Ch}(\mu, f+c) = \mathrm{Ch}^a(\mu, f) + \mathrm{Ch}(\mu, c) = \mathrm{Ch}^a(\mu, f) + I(c). \tag{11}$$

Since any C.M. functional is translatable, that is,

$$I(f+c) = I(f) + I(c), \tag{12}$$

it follows from (11) and (12) that $I(f) = \mathrm{Ch}^a(\mu, f)$.

If $X \in \mathcal{D}$, then $1 \in B(X, \mathcal{D})$, so that I is bounded on $B(X, \mathcal{D})$. Hence we obtain results of Šipoš (1979) and Shimeidler (1986), and the monotone versions of the results of Murofushi et al. (1994) and Révillé (2005).

Theorem 22 (K: 2024)

Let X be a nonempty set and $\mathcal{D}$ a collection of subsets of X with $\emptyset, X \in \mathcal{D}$. Let $I\colon B(X, \mathcal{D}) \to \mathbb{R}$ be a τ-continuous C.M. functional. Then there is a unique finite nonadditive measure μ on 2^X such that

- (a) $I(f) = \mathrm{Ch}^a(\mu, f)$ for every $f \in B(X, \mathcal{D})$,
- (b) μ is inner and outer τ-continuous on $\mathcal{D}$.
- (c) μ is outer $\mathcal{D}$ regular on 2^X.

Note that the representing measure is inner τ-continuous and outer τ-continuous on the same collection $\mathcal{D}$ of sets.