Adder spaces of families of lower continuous functions

Stanisław Kowalczyk

jointly with Małgorzata Turowska

Pomeranian University in Słupsk

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Preliminaries

In the theory of real functions families of functions that are not closed under multiplication and addition are often considered. Therefore for any class of functions X we can search for functions g such that for each $f \in X$, the sum $f + g$ (the product $f \cdot g$) belongs to X. Such a family of functions is called a maximal additive (multiplicative) class. The concept of maximal multiplicative class has been generalized in the following way: given two function classes X and Y we can find all functions g such that the product $f \cdot g$ belongs to X whenever f belongs to Y? Such a family of functions X/Y is called a multiplier of the set X over the set Y

Preliminaries

The original goal of our research was to investigate multipliers over classes of ϱ-lower continuous functions, for different $\varrho \in [0,1]$. ϱ-lower continuity is a kind of generalized continuity. Families of ϱ-lower continuous functions are examples of so called $\mathcal{A}$-continuous functions and path continuous functions with respect to the family $\mathcal{A}$, where $\mathcal{A} = \{A_x \colon x \in \mathbb{R}\}$, where A_x is a some family of "large" subsets of $\mathbb{R}$ containing x. Then continuity of the function $f \colon I \to \mathbb{R}$ at $x \in I$ means that pre-image $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$ contains an element of A_x for every $\varepsilon > 0$. Meanwhile, path continuity at $x \in I$ means that there is a set $E \in A_x$ such that f restricted to E is continuous in ordinary sense at x. Of course, path continuity with respect to $\mathcal{A}$ implies $\mathcal{A}$-continuity, and sometimes they are equivalent. In [3] one can find many examples of generalized continuities defined in this way. In the case of ϱ-lower continuous functions the family $\mathcal{A}$ is defined in terms of the lower density. The density of a measurable set is a very old mathematical concept. Its intensive research began in the 1920's.

The numbers $\underline{d}^+(E,x) = \liminf_{t\to 0^+} \frac{\lambda(E\cap[x,x+t])}{t}$ and $\overline{d}^+(E,x) = \limsup_{t\to 0^+} \frac{\lambda(E\cap[x,x+t])}{t}$ are called the right lower density of E at x and the right upper density of E at x, respectively. The left lower $\underline{d}^-(E,x)$ and the left upper $\overline{d}^-(E,x)$ densities of E at x are defined analogously. If $\underline{d}^+(E,x) = \overline{d}^+(E,x)$ or $\underline{d}^-(E,x) = \overline{d}^-(E,x)$ then we call these numbers the right density or the left density of E at x, respectively. The numbers $\underline{d}(E,x) = \liminf_{t\to 0^+, k\to 0^+} \frac{\lambda(E\cap[x-t,x+k])}{k+t}$ and $\overline{d}(E,x) = \limsup_{t\to 0^+, k\to 0^+} \frac{\lambda(E\cap[x-t,x+k])}{k+t}$ are called the lower and the upper density of E at x, respectively. It is easy to check that $\overline{d}(E,x) = \max\{\overline{d}^+(E,x), \overline{d}^-(E,x)\}$ and $\underline{d}(E,x) = \min\{\underline{d}^+(E,x), \underline{d}^-(E,x)\}$. If $\underline{d}(E,x) = \overline{d}(E,x)$ then we call this number the density of E at x and denote by $d(E,x)$

Density

For every $\varrho \in [0,1)$ let

$$L_\varrho = \{E\colon E \text{ is measurable and } \underline{d}(E,x) > \varrho \text{ for every } x \in E\}.$$

Similarly, for every $\varrho \in (0,1]$ let

$$L_\varrho^\blacklozenge = \{E\colon E \text{ is measurable and } \underline{d}(E,x) \geq \varrho \text{ for every } x \in E\}.$$

ϱ-lower continuity

Definition

Let $\varrho \in [0, 1)$. *We say that* $f : I \to \mathbb{R}$ *is* LC_ϱ*-continuous at* $x \in I$ *if for every* $\varepsilon > 0$ *there exists a measurable* $E \subset \{y \in I : |f(x) - f(y)| < \varepsilon\}$ *such that* $\underline{d}(E, x) > \varrho$. *The set of points at which* f *is* LC_ϱ*-continuous is denoted by* $LC_\varrho(f)$ *and* $\mathcal{LC}_\varrho$ *denotes the set of* $f : I \to \mathbb{R}$ *which are* LC_ϱ*-continuous at every* $x \in I$.

Definition

Let $\varrho \in [0,1)$. We say that $f\colon I \to \mathbb{R}$ is LC_ϱ-continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable $E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ such that $\underline{d}(E,x) > \varrho$. The set of points at which f is LC_ϱ-continuous is denoted by $LC_\varrho(f)$ and $\mathcal{LC}_\varrho$ denotes the set of $f\colon I \to \mathbb{R}$ which are LC_ϱ-continuous at every $x \in I$.

Definition

Let $\varrho \in (0,1]$. We say that $f\colon I \to \mathbb{R}$ is $LC_\varrho^\blacklozenge$-continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable $E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ such that $\underline{d}(E,x) \geq \varrho$. The set of points at which f is $LC_\varrho^\blacklozenge$-continuous is denoted by $LC_\varrho^\blacklozenge(f)$ and $\mathcal{LC}_\varrho^\blacklozenge$ denotes the set of $f\colon I \to \mathbb{R}$ which are $LC_\varrho^\blacklozenge$-continuous at every $x \in I$. In particular, $\mathcal{LC}_1^\blacklozenge$ consists of approximate continuous functions.

When examining properties of multipliers of $\mathcal{LC}_\varrho^\blacklozenge$ and $\mathcal{LC}_\varrho$ families, it turned out that a more appropriate tool is a similarly defined family, which we call the adder of X over Y. Therefore, our work is devoted to determining of adders of $\mathcal{LC}_\varrho^\blacklozenge$ and $\mathcal{LC}_\varrho$ families. As in the case of multipliers, which are a generalization of the maximal multiplicative class, the concept of adder is a generalization of the maximal additive class.

$[\varrho_1, \varrho_2]$-lower superdensity

Now, we introduce the notion of $[\varrho_1, \varrho_2]$-lower superdensity and present its basic properties. To this end, we start with an important theorem showing the equivalence of certain conditions.

Theorem

Let $0 < \varrho_1 \leq \varrho_2 < 1$, $E \subset \mathbb{R}$ be measurable and $x \in E$. The following properties are equivalent:

1. *for every measurable $F \subset \mathbb{R}$ containing x if $\underline{d}(F, x) > \varrho_2$ then $\underline{d}(E \cap F, x) > \varrho_1$;*
2. *for every measurable $F \subset \mathbb{R}$ containing x if $\underline{d}(F, x) \geq \varrho_2$ then $\underline{d}(E \cap F, x) \geq \varrho_1$;*
3. *for every measurable $F \subset \mathbb{R}$ containing x if $\underline{d}(F, x) > \varrho_2$ then $\underline{d}(E \cap F, x) \geq \varrho_1$.*

Definition

Let $0 \le \varrho_1 \le \varrho_2 \le 1$ *and* $\varrho_2 - \varrho_1 < 1$. *For* $0 < \varrho_1 \le \varrho_2 < 1$ *we say that a measurable set* $E \subset \mathbb{R}$ *is* $[\varrho_1, \varrho_2]$*-lower superdense at* $x \in E$ *if it satisfies one of the conditions of the previous theorem. For* $0 \le \varrho_2 < 1$ *we say that a measurable set* $E \subset \mathbb{R}$ *is* $[0, \varrho_2]$*-lower superdense at* $x \in E$ *if it satisfies the first condition of the previous theorem. For* $0 < \varrho_1 \le 1$ *we say that a measurable set* $E \subset \mathbb{R}$ *is* $[\varrho_1, 1]$*-lower superdense at* $x \in E$ *if it satisfies the second condition of the previous theorem. We say that* E *is* $[\varrho_1, \varrho_2]$*-lower superdense if it is* $[\varrho_1, \varrho_2]$*-lower superdense at each of its point. The set of all* $[\varrho_1, \varrho_2]$*-lower superdense subsets of* $\mathbb{R}$ *we denote by* $L(\varrho_1, \varrho_2)$.

Corollary

If E is measurable and $x \in E$ is the density point of E then E is $[\varrho_1, \varrho_2]$-lower superdense at x for every $0 \leq \varrho_1 \leq \varrho_2 \leq 1$ and $\varrho_2 - \varrho_1 < 1$.

$[\varrho_1, \varrho_2]$-lower superdensity

Corollary

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Corollary

Let $0 < \varrho_1 \leq \varrho_2 < 1$ and $E \subset \mathbb{R}$ be measurable. If $E \notin L(\varrho_1, \varrho_2)$ then there exist $x \in E$ and a measurable set $F \subset \mathbb{R}$ such that $\underline{d}(F, x) > \varrho_2$ and $\underline{d}(E \cap F, x) < \varrho_1$.

$[\varrho_1, \varrho_2]$-lower superdensity

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Corollary

Let $0 < \varrho_1 \leq \varrho_2 < 1$ and $E \subset \mathbb{R}$ be measurable. If $E \notin L(\varrho_1, \varrho_2)$ then there exist $x \in E$ and a measurable set $F \subset \mathbb{R}$ such that $\underline{d}(F, x) > \varrho_2$ and $\underline{d}(E \cap F, x) < \varrho_1$.

Theorem

$L^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subset L(\varrho_1, \varrho_2)$ for every $0 \leq \varrho_1 \leq \varrho_2 \leq 1$ and $\varrho_2 - \varrho_1 < 1$. Moreover, $L^{\blacklozenge}_1 = L(\varrho, \varrho)$ for $\varrho \in (0, 1)$ and $L^{\blacklozenge}_\varrho = L(\varrho, 1)$ for $\varrho \in (0, 1]$. Otherwise the inclusion $L^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subset L(\varrho_1, \varrho_2)$ is proper.

$[\varrho_1, \varrho_2]$-lower superdensity

Theorem

Let $0 \leq \varrho_1 \leq \varrho_2 \leq 1$, $\varrho_2 - \varrho_1 < 1$. *Then*

1. $L(\varrho_1, \varrho_2) \subsetneqq U^{\blacklozenge}_{1-\varrho_2+\varrho_1}$;
2. *if* $E \in L(\varrho_1, \varrho_2)$, $x \in E$ *and* $\underline{d}(E, x) = \overline{d}(E, x) = \alpha$ *then* $\alpha \geq 1 - \varrho_2 + \varrho_1$.

$[\varrho_1, \varrho_2]^*$-lower superdensity

Definition

Let $0 \le \varrho_1 < \varrho_2 \le 1$. *A measurable set* $E \subset \mathbb{R}$ *is* $[\varrho_1, \varrho_2]^*$*-lower superdense at* $x \in E$ *if for every measurable* $F \subset \mathbb{R}$ *containing* x *if* $\underline{d}(F, x) \ge \varrho_2$ *then* $\underline{d}(E \cap F, x) > \varrho_1$. *We say that* E *is* $[\varrho_1, \varrho_2]^*$*-lower superdense if it is* $[\varrho_1, \varrho_2]^*$*-lower superdense at each of its point. The set of all* $[\varrho_1, \varrho_2]^*$*-lower superdense subsets of* $\mathbb{R}$ *we denote by* $L^*(\varrho_1, \varrho_2)$.

$[\varrho_1, \varrho_2]^*$-lower superdensity

Definition

Let $0 \le \varrho_1 < \varrho_2 \le 1$. A measurable set $E \subset \mathbb{R}$ is $[\varrho_1, \varrho_2]^*$-lower superdense at $x \in E$ if for every measurable $F \subset \mathbb{R}$ containing x if $\underline{d}(F, x) \ge \varrho_2$ then $\underline{d}(E \cap F, x) > \varrho_1$. We say that E is $[\varrho_1, \varrho_2]^*$-lower superdense if it is $[\varrho_1, \varrho_2]^*$-lower superdense at each of its point. The set of all $[\varrho_1, \varrho_2]^*$-lower superdense subsets of $\mathbb{R}$ we denote by $L^*(\varrho_1, \varrho_2)$.

Theorem

Let $0 \le \varrho_1 < \varrho_2 \le 1$. Then $L_{1-\varrho_2+\varrho_1} \subset L^*(\varrho_1, \varrho_2)$. Moreover, $L_\varrho = L^*(\varrho, 1)$ for $\varrho \in (0, 1)$ and $L_{1-\varrho_2+\varrho_1} \subsetneq L^*(\varrho_1, \varrho_2)$ if $\varrho_2 \ne 1$.

Definition

Let $0 \le \varrho_1 < \varrho_2 \le 1$. *A measurable set* $E \subset \mathbb{R}$ *is* $[\varrho_1, \varrho_2]^*$*-lower superdense at* $x \in E$ *if for every measurable* $F \subset \mathbb{R}$ *containing* x *if* $\underline{d}(F, x) \ge \varrho_2$ *then* $\underline{d}(E \cap F, x) > \varrho_1$. *We say that* E *is* $[\varrho_1, \varrho_2]^*$*-lower superdense if it is* $[\varrho_1, \varrho_2]^*$*-lower superdense at each of its point. The set of all* $[\varrho_1, \varrho_2]^*$*-lower superdense subsets of* $\mathbb{R}$ *we denote by* $L^*(\varrho_1, \varrho_2)$.

Theorem

Let $0 \le \varrho_1 < \varrho_2 \le 1$. *Then* $L_{1-\varrho_2+\varrho_1} \subset L^*(\varrho_1, \varrho_2)$. *Moreover,* $L_\varrho = L^*(\varrho, 1)$ *for* $\varrho \in (0, 1)$ *and* $L_{1-\varrho_2+\varrho_1} \subsetneq L^*(\varrho_1, \varrho_2)$ *if* $\varrho_2 \ne 1$.

Theorem

1. $L^*(\varrho_1, \varrho_2) \subsetneq U_{1-\varrho_2+\varrho_1}$ *for* $0 \le \varrho_1 < \varrho_2 \le 1$;
2. *if* $E \in L^*(\varrho_1, \varrho_2)$, $x \in E$ *and* $\underline{d}(E, x) = \overline{d}(E, x) = \alpha$ *then* $\alpha > 1 - \varrho_2 + \varrho_1$ *for* $0 \le \varrho_1 < \varrho_2 \le 1$.

L_{ϱ_1,ϱ_2}-continuity

Definition

Let $0 \le \varrho_1 \le \varrho_2 \le 1$*,* $\varrho_2 - \varrho_1 < 1$*. We say that* $f\colon I \to \mathbb{R}$ *is* L_{ϱ_1,ϱ_2}*-continuous at* $x \in I$ *if for every* $\varepsilon > 0$ *there exists a measurable* $E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ *which is* $[\varrho_1, \varrho_2]$*-lower superdense at* x*. The set of points at which* f *is* L_{ϱ_1,ϱ_2}*-continuous is denoted by* $L_{\varrho_1,\varrho_2}(f)$ *and* $\mathcal{L}_{\varrho_1,\varrho_2}$ *denotes the set of* $f\colon I \to \mathbb{R}$ *which are* L_{ϱ_1,ϱ_2}*-continuous at every* $x \in I$*.*

L_{ϱ_1,ϱ_2}-continuity

Definition

Let $0 \le \varrho_1 \le \varrho_2 \le 1$, $\varrho_2 - \varrho_1 < 1$. *We say that* $f\colon I \to \mathbb{R}$ *is* L_{ϱ_1,ϱ_2}*-continuous at* $x \in I$ *if for every* $\varepsilon > 0$ *there exists a measurable* $E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ *which is* $[\varrho_1, \varrho_2]$*-lower superdense at* x*. The set of points at which* f *is* L_{ϱ_1,ϱ_2}*-continuous is denoted by* $L_{\varrho_1,\varrho_2}(f)$ *and* $\mathcal{L}_{\varrho_1,\varrho_2}$ *denotes the set of* $f\colon I \to \mathbb{R}$ *which are* L_{ϱ_1,ϱ_2}*-continuous at every* $x \in I$.

Definition

Let $0 \le \varrho_1 < \varrho_2 \le 1$. *We say that* $f\colon I \to \mathbb{R}$ *is* $L^*_{\varrho_1,\varrho_2}$*-continuous at* $x \in I$ *if for every* $\varepsilon > 0$ *there exists a measurable* $E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ *which is* $[\varrho_1, \varrho_2]^*$*-lower superdense at* x*. The set of points at which* f *is* $L^*_{\varrho_1,\varrho_2}$*-continuous is denoted by* $L^*_{\varrho_1,\varrho_2}(f)$ *and* $\mathcal{L}^*_{\varrho_1,\varrho_2}$ *denotes the set of* $f\colon I \to \mathbb{R}$ *which are* $L^*_{\varrho_1,\varrho_2}$*-continuous at every* $x \in I$.

Adder

Definition

Let X and Y be two sets of functions from I to $\mathbb{R}$. We call the set

$$\mathcal{A}(X/Y) = \{g\colon I \to \mathbb{R}\colon f + g \in X \text{ for all } f \in Y\}$$

the adder set of X over Y.

Property of adders

The following properties of $\mathcal{A}(X/Y)$ are immediate consequences of the definition.

Proposition

Let X, X_1, X_2, Y, Y_1, Y_2 be sets of real-valued functions on I. The following statements are true:

1. *If $X_1 \subset X_2$ then $\mathcal{A}(X_1/Y) \subset \mathcal{A}(X_2/Y)$.*
2. *If $Y_1 \subset Y_2$ then $\mathcal{A}(X/Y_2) \subset \mathcal{A}(X/Y_1)$.*
3. *If $0 \in Y$ then $\mathcal{A}(X/Y) \subset X$.*
4. *If $Y \subset X$ and X is closed under addition then $X \subset \mathcal{A}(X/Y)$.*
5. *If $Y \subset X$ and Y is closed under addition then $Y \subset \mathcal{A}(X/Y)$.*
6. *If $0 \in X \cap Y$ and X is closed under addition then $\mathcal{A}(X/Y) = X$ if and only if $Y \subset X$.*

Adders of ϱ-lower continuous functions

Theorem

1. $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)$ *for* $0 < \varrho_1 \leq \varrho_2 \leq 1$,
2. $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$ *for* $0 < \varrho_1 \leq \varrho_2 < 1$,
3. $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$ *for* $0 \leq \varrho_1 \leq \varrho_2 < 1$.

Theorem

1. $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)$ *for* $0 < \varrho_1 \le \varrho_2 \le 1$*,*
2. $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$ *for* $0 < \varrho_1 \le \varrho_2 < 1$*,*
3. $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$ *for* $0 \le \varrho_1 \le \varrho_2 < 1$*.*

Corollary

1. $\mathcal{LC}^{\blacklozenge}_{1} = \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho}/\mathcal{LC}^{\blacklozenge}_{\varrho}\right)$ *for* $\varrho \in (0,1]$, $\mathcal{LC}^{\blacklozenge}_{\varrho} = \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho}/\mathcal{LC}^{\blacklozenge}_{1}\right)$ *for* $\varrho \in (0,1)$ *and* $\mathcal{LC}^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subsetneqq \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)$ *for* $0 < \varrho_1 < \varrho_2 < 1$*,*
2. $\mathcal{LC}^{\blacklozenge}_{1} = \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho}/\mathcal{LC}_{\varrho}\right)$ *for* $\varrho \in (0,1)$ *and* $\mathcal{LC}^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subsetneqq \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$ *for* $0 < \varrho_1 < \varrho_2 < 1$*,*
3. $\mathcal{LC}^{\blacklozenge}_{1} = \mathcal{A}\left(\mathcal{LC}_{\varrho}/\mathcal{LC}_{\varrho}\right)$ *for* $\varrho \in (0,1]$ *and* $\mathcal{LC}^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subsetneqq \mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$ *for* $0 \le \varrho_1 < \varrho_2 < 1$*.*

Adders of ϱ-lower continuous functions

Theorem

Let $0 \le \varrho_1 < \varrho_2 \le 1$. *Then* $\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right) = \mathcal{L}^*_{\varrho_1,\varrho_2}$.

Theorem

Let $0 \leq \varrho_1 < \varrho_2 \leq 1$. *Then* $\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right) = \mathcal{L}^*_{\varrho_1,\varrho_2}$.

Corollary

$\mathcal{LC}_{\varrho} = \mathcal{A}\left(\mathcal{LC}_{\varrho}/\mathcal{LC}^{\blacklozenge}_{1}\right)$ *for* $\varrho \in [0,1)$ *and*
$\mathcal{LC}_{1-\varrho_2+\varrho_1} \subsetneq \mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)$ *for* $0 \leq \varrho_1 < \varrho_2 < 1$.

Adders of ϱ-lower continuous functions

Theorem

Let $0 \leq \varrho_1 < \varrho_2 \leq 1$. *Then* $\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right) = \mathcal{L}^{*}_{\varrho_1,\varrho_2}$.

Corollary

$\mathcal{LC}_{\varrho} = \mathcal{A}\left(\mathcal{LC}_{\varrho}/\mathcal{LC}^{\blacklozenge}_{1}\right)$ *for* $\varrho \in [0,1)$ *and*
$\mathcal{LC}_{1-\varrho_2+\varrho_1} \subsetneqq \mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)$ *for* $0 \leq \varrho_1 < \varrho_2 < 1$.

Remark

As we mentioned before, multipliers of ϱ*-lower continuous functions have more complicated structure than adders. For example, for* $0 < \varrho_1 < \varrho_2 < 1$ *we have* $\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right) = \mathcal{L}_{\varrho_1,\varrho_2} \subsetneqq \mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}$.

Path ϱ-lower continuity

Definition

Let $\varrho \in [0,1)$. We say that $f\colon I \to \mathbb{R}$ is LPC_ϱ-continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable $E_\varepsilon \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ such that $\lim_{\varepsilon\to 0^+} \underline{d}(E_\varepsilon, x) > \varrho$. The set of points at which f is LPC_ϱ-continuous is denoted by $LPC_\varrho(f)$ and $\mathcal{LPC}_\varrho$ denotes the set of $f\colon I \to \mathbb{R}$ for which are LPC_ϱ-continuous at every $x \in I$.

Path ϱ-lower continuity

Definition

Let $\varrho \in [0,1)$. We say that $f\colon I \to \mathbb{R}$ is LPC_ϱ-continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable $E_\varepsilon \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ such that $\lim_{\varepsilon \to 0^+} \underline{d}(E_\varepsilon, x) > \varrho$. The set of points at which f is LPC_ϱ-continuous is denoted by $LPC_\varrho(f)$ and $\mathcal{LPC}_\varrho$ denotes the set of $f\colon I \to \mathbb{R}$ for which are LPC_ϱ-continuous at every $x \in I$.

Theorem

Let $\varrho \in (0,1]$, $f\colon I \to \mathbb{R}$ and $x \in I$. Then f is LPC_ϱ-continuous at $x \in I$ if and only if there exist a measurable set $E \subset I$ such that $x \in E$, f restricted to E is continuous at x and $\underline{d}(E, x) > \varrho$.

Adders of path LPC_ϱ-continuous functions

Theorem

a) $\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) = \mathcal{L}_{\varrho_1,\varrho_2}$ *for every* $0 < \varrho_1 \leq \varrho_2 < 1$,

b) $\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) = \mathcal{L}_{\varrho_1,\varrho_2}$ *for every* $0 \leq \varrho_1 \leq \varrho_2 < 1$.

Theorem

a) $\mathcal{A}\left(\mathcal{LC}^\blacklozenge_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) = \mathcal{L}_{\varrho_1,\varrho_2}$ *for every* $0 < \varrho_1 \leq \varrho_2 < 1$,

b) $\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) = \mathcal{L}_{\varrho_1,\varrho_2}$ *for every* $0 \leq \varrho_1 \leq \varrho_2 < 1$.

Theorem

a) $\bigcup_{\varrho\in(\varrho_1,\varrho_2)} \mathcal{L}_{\varrho,\varrho_2} \subset \mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}^\blacklozenge_{\varrho_2}\right) \subset \mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right) \subset \mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) \subset \mathcal{L}_{\varrho_1,\varrho_2}$ *for every* $0 \leq \varrho_1 < \varrho_2 < 1$,

b) $\mathcal{A}\left(\mathcal{LPC}_{\varrho}/\mathcal{LPC}_{\varrho}\right) = \mathcal{L}_{\varrho,\varrho} = \mathcal{LC}^\blacklozenge_1$ *and* $\mathcal{A}\left(\mathcal{LPC}_{\varrho}/\mathcal{LC}^\blacklozenge_1\right) = \mathcal{LPC}_\varrho$ *for* $\varrho \in [0, 1)$.

THANK YOU

FOR YOUR ATTENTION

References I

[1] J. Appell, P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Univ. Press 1990.

[2] M. Bienias, S. Głąb, W. Wilczyński, *Cardinality of sets of ϱ-upper and ϱ-lower continuous functions*, Bull. Soc. Sci. Lett. Łódź Ser. Rech. Deform. 64 (2014), 71–80.

[3] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Math., Vol. 659, Springer-Verlag, New York, 1978.

[4] A. M. Bruckner, R. J. O'Malley, B. S. Thomson, *Path Derivatives: A Unified View of Certain Generalized Derivatives*, Trans. Amer. Math. Soc. 283 (1984), 97–125.

[5] D. Bugajewska, S. Reinwand, *Some Remarks on Multiplier Spaces 1: Classical Spaces*, Z. Anal. Anwend. 38(2)(2019), 125–142.

[6] D. Bugajewska, S. Reinwand, *Some Remarks on Multiplier Spaces 2: BV-type Spaces*, Z. Anal. Anwend. 38(3)(2019), 309–327.

[7] T. Filipczak, *On some abstract density topologies*, Real Anal. Exchange 14 (1988/89), 140–166.

[8] R. J. Fleissner, *Distant bounded variation and products of derivatives*, Fund. Math. 94 (1977), 1–11.

[9] A. Karasińska, E. Wagner-Bojakowska, *Some remarks on ϱ-upper density*, Tatra Mt. Math. Publ. 46 (2010), 85–89.

References II

[10] S. Kowalczyk, K. Nowakowska, *Maximal classes for the family of* $[\lambda, \varrho]$*-continuous function*, Real Anal. Exchange, Vol. 37(1), 2011/2012, 307–324.

[11] S. Kowalczyk, M. Turowska, *Topologies on normed spaces generated by porosity*, Filomat 33:1 (2019), 335–352.

[12] S. Kowalczyk, M. Turowska, *Topologies generated by porosity and maximal additive and multiplicative families for porouscontinuous functions*, Topology Appl. 239 (2018) 1–13.

[13] S. Kowalczyk, M. Turowska, *On topologies generated by lower porosity*, Results Math (2022) 77:220.

[14] S. Kowalczyk, M. Turowska, *Path continuity connected with density and porosity*, Modern real analysis, Wydawnictwo Uniwersytetu Łódzkiego 2015, 105–126.

[15] J. Mařík, *Multipliers of summable derivatives*, Real Anal. Exchange 8 (1982), 486–493.

[16] V. Maz'ya, T. O. Shaposhnikova, *Theory of Sobolev Multipliers with Applications to Differential and Integral Operators*, Grundlehren Math. Wiss. 337. Berlin: Springer 2009.

[17] R. O'Neil, *Fractional integration in Orlicz spaces*. Trans. Amer. Math. Soc. 115 (1965), 300–328.

[18] Th. Radakovic, *Über Darbouxsche und stetige Funktionen* (in German). Monatsh. Math. Phys. 38 (1931), 117–122.

[19] S. Reinwand, P. Kasprzak, *Multiplication operators in BV spaces*, Annali di Matematica Pura ed Applicata, 202(2)(2023), 787–826.

[20] D. N. Sarkhel, A. K. De, *The proximally continuous integrals*, J. Aust. Math. Soc. (A) 31 (1981), 26–45.

[21] W. Wilczyński, *Density topologies*, Handbook of Measure Theory, chapter 15, Elsevier 2012, 307–324.

[22] W. Wilkosz, *Some properties of derivative functions*, Fund. Math. 2 (1921), 145–154.

Thank You
for your attention !!!