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Chaotic and antichaotic families of functions

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Let us adopt the notation:

$$G^0 = \{\mathrm{id}_X\} \text{ and } G^n = \{g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_n} : i_1, i_2, \ldots, i_n \in [\![0, k]\!]\} \text{ for } n \in \mathbb{N}.$$

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For $f : X \to X$ let us put

$$G_f^0 = \{\mathrm{id}_X\} \text{ and } G_f^n = \{g \circ f : g \in G^{n-1}\} \cup \{\mathrm{id}_X\} \text{ for } n \in \mathbb{N}.$$

Obviously

$$G_f^0 \subset G_f^1 \subset G_f^2 \subset \dots$$

An entropy of f with respect to the family G

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Let $G = \{g_0 = \mathrm{id}_X, g_1, \ldots, g_k\} \subset X^X$, $n \in \mathbb{N}$, $\varepsilon > 0$, $f : X \to X$ and $Y \subset X$ be a non-empty set.

We shall say that $Z \subset Y$ is an (n, ε, G, Y)-separated set of f in Y by G if for any two different points $p, q \in Z$ there exists a function $\xi \in G_f^n$ such that $\rho(\xi(p), \xi(q)) > \varepsilon$.

An entropy of f with respect to the family G in Y is the number

$$h(f, G, Y) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_f(n, \varepsilon, G, Y),$$

where $s_f(n, \varepsilon, G, Y)$ denotes the maximal cardinality of (n, ε, G, Y)-separated set of f in Y by G.

We write briefly $h(f, G)$ if $Y = X$.

Comparison with the entropy of a semigroup

Comparison with the entropy of a semigroup

Let $G = \{g_0 = \mathrm{id}_X, g_1, \ldots, g_k\} \subset X^X$ be a set of generators and $\mathcal{G}(G) = \bigcup_{n \in \mathbb{N}} G^n$ (where $G^n = \{g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_n} : i_1, i_2, \ldots, i_n \in [\![0, k]\!]\}$).

Let $n \in \mathbb{N}$, $\varepsilon > 0$ and $Y \subset X$. We say that $Z \subset Y$ is (n, ε)*-separated by* $\mathcal{G}(G)$ *in* Y if for any two distinct points $p, q \in Z$ there exists a function $g \in G^n$ such that $\rho(g(p), g(q)) > \varepsilon$.

Let $s(n, \varepsilon, \mathcal{G}(G), Y)$ denote the maximal cardinality of (n, ε)-separated set by $\mathcal{G}(G)$ in Y. Then the entropy of a semigroup $\mathcal{G}(G)$ on Y is the number:

$$h(\mathcal{G}(G), Y) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \mathcal{G}(G), Y).$$

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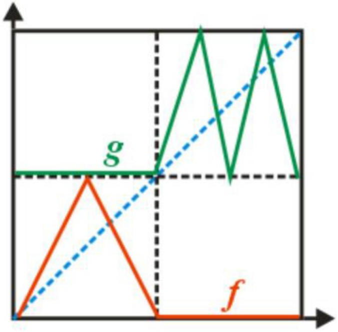
Property

If $f \in G$ *and* $Y \subset X$ *then* $h(f, G, Y) \leqslant h(\mathcal{G}(G), Y)$.

Let $f, g : [0,1] \to [0,1]$ be defined as follows:

$$f(x) = \begin{cases} 2x & \text{for } x \in \left[0, \frac{1}{4}\right] \\ -2x+1 & \text{for } x \in \left(\frac{1}{4}, \frac{1}{2}\right) \\ 0 & \text{for } x \in \left[\frac{1}{2}, 1\right] \end{cases} \qquad g(x) = \begin{cases} \frac{1}{2} & \text{for } x \in \left[0, \frac{1}{2}\right] \\ 4x - \frac{3}{2} & \text{for } x \in \left(\frac{1}{2}, \frac{5}{8}\right] \\ -4x + \frac{7}{2} & \text{for } x \in \left(\frac{5}{8}, \frac{3}{4}\right] \\ -4x + \frac{9}{2} & \text{for } x \in \left(\frac{7}{8}, 1\right] \end{cases}$$



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$$G = \{\mathrm{id}_{[0,1]}, f, g\}$$

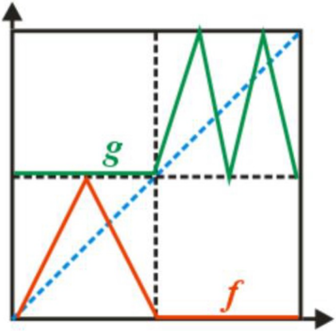
$$G^n = \{\mathrm{id}_{[0,1]}, \mathrm{const}_{\frac{1}{2}}, \mathrm{const}_0\} \cup \{f^p : p \in [\![1,n]\!]\} \cup \{g^p : p \in [\![1,n]\!]\}$$

$$G_f^n = \{\mathrm{id}_{[0,1]}, \mathrm{const}_{\frac{1}{2}}\} \cup \{f^p : p \in [\![1,n]\!]\}$$

$$\log 4 = h(g) \leqslant h(\mathcal{G}(G))$$

$$h(f, G) = h(f) = \log 2$$

$$h(f, G) < h(\mathcal{G}(G)).$$



Comparison with the entropy of a function

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Let $\emptyset \neq Y \subset X$ and $G = \{g_0, g_1\} \subset X^X$ be a set of generators such that $g_0 = \mathrm{id}_X$ and $g_1 = f$, then

$$h(f, G, Y) = h(f, Y).$$

Chaotic and antichaotic families of functions

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We will say that a finite family of functions $G = \{g_0 = \mathrm{id}_X, g_1, \ldots, g_k\}$ is antichaotic on the set $\emptyset \neq Y \subset X$, if $h(g_i, Y) = 0$ for $i \in [\![0, k]\!]$. Obviously if $X = Y$ we will say that G is antichaotic.

We will say that a finite family of functions $G = \{g_0 = \mathrm{id}_X, g_1, \ldots, g_k\}$ is chaotic on the set $\emptyset \neq Y \subset X$, if $h(g_i, Y) > 0$ for $i \in [\![1, k]\!]$. Obviously if $X = Y$ we will say that G is chaotic.

$C(X)$ – the space of all continuous functions $f : X \to X$.
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Let consider subspace $C_V^*(X)$ composed of sets of the following form $\{g_1, \ldots, g_k\}$ for $g_i \in C(X)$ and $k \in \mathbb{N}$.

Let $C_V^{*0}(X)$ denote the family of all sets $\{g_1, \ldots, g_k\} \in C_V^*(X)$ such that the family $\{g_0 = \mathrm{id}_X, g_1, \ldots, g_k\}$ is antichaotic.

Let $C_V^{*+}(X)$ denote the family of all sets $\{g_1, \ldots, g_k\} \in C_V^*(X)$ such that $\{g_0 = \mathrm{id}_X, g_1, \ldots, g_k\}$ is chaotic.

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Theorem

Let $X = [0, 1]$. The set $\mathrm{Int}_H(C_V^{*+}(X))$ is dense in $C_V^*(X)$.

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Theorem

Let $X = [0, 1]$. The set $\mathrm{Int}_H(C_V^{*+}(X))$ is dense in $C_V^*(X)$.

Corollary

The set $C_V^{*0}([0, 1])$ is nowhere dense in $C_V^*([0, 1])$.

Theorem

Let (X, ρ) be a connected topological manifold having the fixed point property. The set $C_V^{*+}(X)$ is dense in $C_V^*(X)$.

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Corollary

Let (X,ρ) be a connected topological manifold having the fixed point property. Then $\mathrm{Int}_H(C_V^{*0}(X)) = \emptyset$.

α-entropy points and chaos

Let $G = \{g_0 = \mathrm{id}_X, g_1, \ldots, g_k\} \subset X^X$ and $\alpha \geqslant 1$. We say that $x_0 \in X$ is an α-entropy point of $f : X \to X$ by G if $h(f, G, Y) \geqslant \log \alpha$ for any open neighborhood Y of the point x_0.

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So, if there exists an open neighborhood Y_0 of point x_0 such that $h(f, G, Y_0) = 0 = \log 1$ then we cannot talk about chaos.

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If $\alpha > 1$ and for any open neighborhood Y of point x_0 we have $h(f, G, Y) \geqslant \log \alpha$ then we will call x_0 an α-chaotic point of f by G.

Theorem

Let $\alpha > 1$, $x_0 \in [0,1)$ and $f : [0,1] \to [0,1]$ be the Darboux function such that $x_0 \in \mathrm{Fix}(f)$ and $D^+f(x_0) > 0$. Then there exists antichaotic family $G \subset C([0,1])$ such that $\#(G) = 2 \cdot \lceil \alpha \rceil + 1$ and x_0 is an α-chaotic point of f by G.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined in a neighborhood of x_0. We define the *upper right Dini derivative* D^+f of f at x_0 by

$$D^+f(x_0) = \limsup_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h}.$$

$\lceil \alpha \rceil = \min\{k \in \mathbb{N}_0 : k \geqslant \alpha\}$

$x_0 \in \mathrm{Fix}(f)$ iff $f(x_0) = x_0$

Theorem

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Problem

To prove the theorem above, we need to consider a family G containing $2 \cdot \lceil \alpha \rceil + 1$ elements. The question arises whether (with possibly stronger assumptions regarding the function f) there is a family G with fewer elements such that the conclusion of the above theorem is satisfied?

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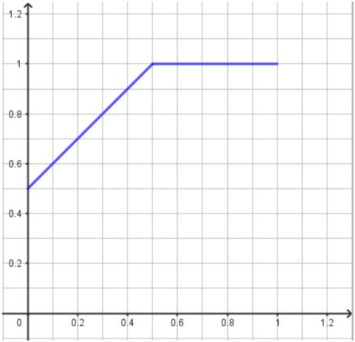
Remark

In the context of above theorem the opposite situation can be considered. For a fixed (antichaotic) family $G = \{g_0 = \mathrm{id}_{[0,1]}, g_1, \ldots, g_k\} \subset C([0,1])$, can we find the function $f : [0,1] \to [0,1]$ such that for every/some $\alpha > 1$ there exists x_0 which is the α-chaotic point of f by G?

Let $G = \{g_0 = \mathrm{id}_{[0,1]}, g_1\}$, where g_1 is a function given by:

$$g_1(x) = \begin{cases} x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$



We get, that $g_1 \circ g_1$ is a constant function. Consequently, for any $\varepsilon > 0$ and $n > 2$ we have $s_f(n, \varepsilon, G) = s_f(2, \varepsilon, G)$ for any function f, which means that $h(f, G) = 0$.

Disruption

Let $f \in C([0,1])$ and $G = \{g_0 = \mathrm{id}_{[0,1]}, g_1, ..., g_k\} \subset C([0,1])$.

We shall say that the family $G_f = \{g_0 = \mathrm{id}_{[0,1]}, f \circ g_1, ..., f \circ g_k\}$ is an f-disruption of G.

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Theorem

Let $G = \{g_0 = \mathrm{id}_{[0,1]}, g_1, \ldots, g_k\}$ be a family of continuous functions mapping $[0,1]$ into itself. Moreover, there exist $x_0 \in \mathrm{Fix}(G) \cap [0,1)$ and $\delta > 0$ such that $(g_i)'_+(x) > 1$ for any $i \in [\![1,k]\!]$ and $x \in [x_0, x_0+\delta]$. Then for any $\alpha > 1$ there exists a function $f \in C([0,1])$ such that the family G_f being an f-disruption of G is chaotic and x_0 is α-chaotic point of f by G_f.

- L. Alsedá, J. Llibre, M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, World Sci., 2000 (second edition).
- A. Biś, *Entropies of a semigroup of maps*, Discrete Contin. Dynam. Systems 11(2&3), 2004, 639–648.
- A. Biś, M. Urbański, *Some remarks on topological entropy of a semigroup of continuous maps*, Cubo 8, 2006, 63–71.
- A.M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics, vol. 659 Springer-Verlag Berlin Heidelberg 1978.
- E. Korczak-Kubiak, A. Loranty, R.J. Pawlak, *Measuring chaos by entropy for a finite family of functions*, Chaos 30, 063138, 2020.
- S. Kolyada, L. Snoha, *Topological entropy of nonautonomous dynamical systems. Random & Computational Dynamics* 1996, *4 (2 & 3)*, 205–233.
- D. Kwietniak and P. Oprocha, *Topological entropy and chaos for maps induced on hyperspaces*, Chaos, Solitons & Fractals. 33, 2007, 76–86.
- R. J. Pawlak, J. Poprawa, *On some dense sets in the space of dynamical systems*, Advanced Nonlinear Studies. 23 (1), 2023, pp. 20220053;
- R. J. Pawlak, J. Poprawa, *On disruptions of semigroups and points strongly focusing chaos*, in preparation.