Mixing of linear operators on Banach spaces with respect to infinitely divisible measures

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18 Jun 2024

Dynamical systems and mixing

Definition 1

A *measure-preserving dynamical system* $(E, \mathcal{B}, \mu, T)$ is a probability space $(E, \mathcal{B}, \mu)$ with a measure-preserving transformation T on it.

In the sequel, we let E be a complex Banach space.

Definition 2

We say that $(E, \mathcal{B}, \mu, T)$ is *mixing* if

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B), \qquad \forall A, B \in \mathcal{B},$$

and *weak mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\mu(A \cap T^{-k}B) - \mu(A)\mu(B)\right| = 0, \qquad \forall A, B \in \mathcal{B}.$$

A brief history of investigations

	Stochastic processes	**Hilbert and Banach spaces**
Gaussian	Wiener and Akutowicz ([WA57])	Flytzanis ([Fly95]) Bayart et al. (e.g. [BG06, BM09, BM16])
Infinitely divisible	Rosiński and Żak ([RŻ96, RŻ97]) Fuchs and Stelzer ([FS13]) Passeggeri and Veraart ([PV19])	Mau and Privault ([MP24])

Various other dynamical systems: Cornfeld et al. ([CFS82])

Gaussian measures and mixing on Banach spaces

A probability measure μ on E is Gaussian if it has characteristic functional

$$\int_E e^{i\,\mathrm{Re}\langle z,x^*\rangle}\mu(dz) = \exp\left(-\frac{1}{4}\langle Rx^*,x^*\rangle\right), \qquad x^* \in E^*,$$

where R is defined by

$$\langle Rx^*,y^*\rangle = \int_E \overline{\langle z,x^*\rangle}\langle z,y^*\rangle\mu(dz), \qquad x^*,y^* \in E^*,$$

and is called a covariance operator, see e.g. [BM09].

Theorem 3 ([BM09])

Let $(E,\mathcal{B},\mu,T)$ be given with E a complex separable Banach space and μ a Gaussian measure with full support and covariance operator R. Then, $(E,\mathcal{B},\mu,T)$ is

1. *mixing if and only if $\lim_{n\to\infty}\langle RT^{*n}x^*,y^*\rangle = 0$ for all $x^*,y^* \in E^*$, and*
2. *weak mixing if and only if $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|\langle RT^{*n}x^*,y^*\rangle| = 0$ for all $x^*,y^* \in E^*$.*

Infinite divisibility

Definition 4

A probability measure μ is said to be infinitely divisible if for every n there exists a probability measure μ_n such that $\mu = (\mu_n)^n$.

The characteristic functional of any infinitely divisible probability measure μ on E can be written as

$$\int_E e^{i \operatorname{Re}\langle z, x^* \rangle} \mu(dz)$$
$$= \exp\left(-\frac{1}{4} \langle Rx^*, x^* \rangle + \int_E \left(e^{i \operatorname{Re}\langle z, x^* \rangle} - 1 - ik(z) \operatorname{Re} \langle z, x^* \rangle \right) \lambda(dz) \right), \quad x^* \in E^*,$$

where

- $k(z)$ is a bounded measurable function on E such that $\lim_{z \to 0} k(z) = 1$ and $k(z) = O(1/\|z\|)$ as $\|z\| \to \infty$, called a truncation function, and
- λ is a Lévy measure, i.e. λ is a measure on E that satisfies $\lambda(\{0\}) = 0$ and

$$\int_E \min((\operatorname{Re} \langle z, x^* \rangle)^2, 1) \lambda(dz) < \infty, \qquad x^* \in E^*.$$

Codifference functionals

Our problem

We desire to derive mixing criteria for the more general class of infinitely divisible measures.

Definition 5

Let X be an E-valued random variable with distribution μ. We define for all $x^*, y^* \in E^*$, the *codifference functionals*

$$C_\mu^{=}(x^*, y^*) := \log \mathbb{E}\left[e^{i\,\mathrm{Re}\left\langle X, x^* - y^*\right\rangle}\right] - \log \mathbb{E}\left[e^{i\,\mathrm{Re}\left\langle X, x^*\right\rangle}\right] - \log \mathbb{E}\left[e^{-i\,\mathrm{Re}\left\langle X, y^*\right\rangle}\right],$$

$$C_\mu^{\neq}(x^*, y^*) := \log \mathbb{E}\left[e^{i\,\mathrm{Re}\left\langle X, x^*\right\rangle - i\,\mathrm{Im}\left\langle X, y^*\right\rangle}\right] - \log \mathbb{E}\left[e^{i\,\mathrm{Re}\left\langle X, x^*\right\rangle}\right] - \log \mathbb{E}\left[e^{-i\,\mathrm{Im}\left\langle X, y^*\right\rangle}\right].$$

If μ is Gaussian we have

$$C_\mu^{=}(x^*, y^*) = \frac{1}{2}\,\mathrm{Re}\,\langle Rx^*, y^*\rangle \qquad \text{and} \qquad C_\mu^{\neq}(x^*, y^*) = \frac{1}{2}\,\mathrm{Im}\,\langle Rx^*, y^*\rangle.$$

A mixing bridge

In the sequel, we let X denote a random variable with distribution μ on E.

Lemma 6 ([MP24])

A bounded linear operator $T : E \to E$ is mixing (resp. weakly mixing) with respect to μ if and only if the $\mathbb{R}^2$-valued process defined as

$$\big(\mathrm{Re}\langle X, T^{*n}x^*\rangle, \mathrm{Im}\langle X, T^{*n}x^*\rangle\big), \qquad n \geq 0,$$

induced by X is mixing (resp. weakly mixing) for every $x^* \in E^*$.

Note: The process is mixing in the following sense:

- The space is $(\mathbb{R}^2)^{\mathbb{N}}$.
- The probability measure is the joint distribution.
- The operator is the forward shift operator by one time step.

Mixing and weak mixing for $\mathbb{R}^2$-valued processes, part 1

Theorem 7 ([PV19])

Let $(X_t)_{t\in\mathbb{N}}$ *be an* $\mathbb{R}^2$*-valued stationary infinitely divisible process such that* ν_0*, the Lévy measure of* X_0*, satisfies*

$$\nu_0\{(x_1,x_2)\in\mathbb{R}^2 : \exists j\in\{1,2\}, x_j\in 2\pi\mathbb{Z}\}=0.$$

Then $(X_t)_{t\in\mathbb{N}}$ *is mixing if and only if for any* $j,k=1,2$ *we have*

$$\lim_{t\to\infty}\mathbb{E}\left[e^{i(X_t^{(j)}-X_0^{(k)})}\right]=\mathbb{E}\left[e^{iX_0^{(j)}}\right]\mathbb{E}\left[e^{-iX_0^{(k)}}\right],$$

and weakly mixing if and only if there exists a density one set $D\subset\mathbb{N}$ *we have for any* $j,k=1,2$*,*

$$\lim_{n\to\infty,n\in D}\mathbb{E}\left[e^{i(X_t^{(j)}-X_0^{(k)})}\right]=\mathbb{E}\left[e^{iX_0^{(j)}}\right]\mathbb{E}\left[e^{-iX_0^{(k)}}\right]$$

where $X_t^{(j)}$ *denotes the jth component of* X_t*.*

Mixing and weak mixing for infinite divisibility, part 1

Proposition 8 ([MP24])

Let E be a complex Banach space, and assume that for every $x^* \in E^*$, the Lévy measure ν_{x^*} of $(\operatorname{Re}\langle X, x^*\rangle, \operatorname{Im}\langle X, x^*\rangle)$ satisfies

$$\nu_{x^*}\{(x_1, x_2) \in \mathbb{R}^2 : \exists j \in \{1,2\}, x_j \in 2\pi\mathbb{Z}\} = 0.$$

Then, a bounded linear operator $T : E \to E$ that leaves invariant the infinitely divisible measure μ is strongly mixing with respect to μ if and only if

$$\lim_{n\to\infty} C_\mu^{=}\left(x^*, T^{*n}x^*\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} C_\mu^{\neq}\left(x^*, T^{*n}x^*\right) = 0, \quad x^* \in E^*,$$

and weakly mixing if and only if there exists a density one subset D_{x^*} of $\mathbb{N}$ such that

$$\lim_{\substack{n\to\infty \\ n\in D_{x^*}}} \frac{1}{n}\sum_{k=0}^{n-1} \left|C_\mu^{=}(x^*, T^{*k}x^*)\right| = 0 \quad \text{and} \quad \lim_{\substack{n\to\infty \\ n\in D_{x^*}}} \frac{1}{n}\sum_{k=0}^{n-1} \left|C_\mu^{\neq}(x^*, T^{*k}x^*)\right| = 0, \quad x^* \in E^*.$$

Interlude: A technical condition

The requirement

$$\nu_{x^*}\{(x_1, x_2) \in \mathbb{R}^2 : \exists j \in \{1,2\}, x_j \in 2\pi\mathbb{Z}\} = 0$$

is a technical condition from [PV19]. In the case

$$\nu_{x^*}\{(x_1, x_2) \in \mathbb{R}^2 : \exists j \in \{1,2\}, x_j \in 2\pi\mathbb{Z}\} > 0,$$

we can argue on a dilation of the process instead.

Definition 9

For a Lévy measure ν on $\mathbb{R}^2$, let A be the collection of values $s \in \mathbb{R}$ such that there exists $j \in \{1,2\}$ such that

$$\nu\left(\{x = (x_1, x_2) \in \mathbb{R}^2 : x_j = s\}\right) > 0.$$

We define the set

$$Z = \{2\pi k/s : k \in \mathbb{Z}, s \in A\}.$$

Then $\mathbb{R} \backslash Z$ is non-empty.

Mixing and weak mixing for $\mathbb{R}^2$-valued processes, part 2

Theorem 10

Let $(X_t)_{t\in\mathbb{N}}$ be an $\mathbb{R}^2$-valued stationary infinitely divisible process such that ν_0, the Lévy measure of X_0, satisfies

$$\nu_0\{(x_1, x_2) \in \mathbb{R}^2 : \exists j \in \{1, 2\}, x_j \in 2\pi\mathbb{Z}\} > 0.$$

To ν_0 associate the set Z. Then $(X_t)_{t\in\mathbb{N}}$ is mixing if and only if for some non-zero $a \in \mathbb{R}\backslash Z$, for any $j, k = 1, 2$ we have

$$\lim_{t\to\infty} \mathbb{E}\left[e^{ia(X_t^{(j)}-X_0^{(k)})}\right] = \mathbb{E}\left[e^{iaX_0^{(j)}}\right]\mathbb{E}\left[e^{-iaX_0^{(k)}}\right],$$

and weakly mixing if and only if for some non-zero $a \in \mathbb{R}\backslash Z$, and density one set $D \subset \mathbb{N}$, for any $j, k = 1, 2$ we have

$$\lim_{n\to\infty, n\in D} \mathbb{E}\left[e^{ia(X_t^{(j)}-X_0^{(k)})}\right] = \mathbb{E}\left[e^{iaX_0^{(j)}}\right]\mathbb{E}\left[e^{-iaX_0^{(k)}}\right]$$

where $X_t^{(j)}$ denotes the jth component of X_t.

Mixing and weak mixing for infinite divisibility, part 2

Proposition 11

Let E be a complex Banach space, and assume that for every $x^* \in E^*$, the Lévy measure ν_{x^*} of $(\mathrm{Re}\langle X, x^*\rangle, \mathrm{Im}\langle X, x^*\rangle)$ *satisfies*

$$\nu_{x^*}\{(x_1, x_2) \in \mathbb{R}^2 : \exists j \in \{1,2\}, x_j \in 2\pi\mathbb{Z}\} > 0.$$

To each ν_{x^*} associate the set Z_{x^*}. Then, a bounded linear operator $T : E \to E$ that leaves invariant the infinitely divisible measure μ is strongly mixing with respect to μ if and only if for some non-zero $a \in \mathbb{R}\backslash Z_{x^*}$,

$$\lim_{n\to\infty} C_\mu^{=}\left(ax^*, aT^{*n}x^*\right) = 0 \quad \textit{and} \quad \lim_{n\to\infty} C_\mu^{\neq}\left(ax^*, aT^{*n}x^*\right) = 0, \quad x^* \in E^*,$$

and weakly mixing if and only if for some non-zero $a \in \mathbb{R}\backslash Z_{x^*}$ and density one subset D_{x^*} of $\mathbb{N}$ we have

$$\lim_{\substack{n\to\infty \\ n\in D_{x^*}}} \frac{1}{n}\sum_{k=0}^{n-1}\left|C_\mu^{=}(ax^*, aT^{*k}x^*)\right| = 0 \quad \textit{and} \quad \lim_{\substack{n\to\infty \\ n\in D_{x^*}}} \frac{1}{n}\sum_{k=0}^{n-1}\left|C_\mu^{\neq}(ax^*, aT^{*k}x^*)\right| = 0,$$

$x^* \in E^*$.

Mixing criteria via stochastic integrals

We now represent X with distribution μ as the stochastic integral $X := \int_E z\Lambda(dz)$ where Λ is an independently scattered infinitely divisible random measure on E. The characteristic functional of X can be written as

$$\int_E e^{i\,\mathrm{Re}\langle z,x^*\rangle}\mu(dz)$$
$$= \exp\left(-\frac{1}{4}\langle Rx^*,x^*\rangle + \int_E\int_{-\infty}^{\infty}\left(e^{iu\,\mathrm{Re}\langle z,x^*\rangle} - 1 - iuk(u)\,\mathrm{Re}\langle z,x^*\rangle\right)\rho(z,du)\xi(dz)\right),$$

$x^* \in E^*$, where

- $\sigma^2 : E \to [0,\infty)$ is a measurable function,
- $\langle Rx^*,y^*\rangle = \int_E \overline{\langle z,x^*\rangle}\,\langle z,y^*\rangle\,\sigma^2(z)\xi(dz)$,
- $\{\rho(s,\cdot)\}_{s\in E}$ is a family of Lévy measures on $\mathbb{R}$,
- $k(u) = 1$ if $|u| < 1$, and $k(u) = 1/|u|$ otherwise, and
- ξ is a σ-finite measure called a control measure.

Mixing criteria via stochastic integrals

We let

$$V(r, z) := \int_{-\infty}^{\infty} \min(|ru|, 1)\rho(z, du), \qquad r \in \mathbb{R},\ z \in E.$$

Theorem 12 ([MP24])

A bounded linear operator $T : E \to E$ that leaves invariant the infinitely divisible measure μ is strongly mixing with respect to μ if and only if for every $x^* \in E^*$ we have

$$\lim_{n\to\infty} \langle RT^{*n}x^*, x^* \rangle = 0,$$

$$\lim_{n\to\infty} \int_E V\big(|\operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle|, z\big)\xi(dz) = 0,$$

and

$$\lim_{n\to\infty} \int_E V\big(|\operatorname{Re}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle|, z\big)\xi(dz) = 0.$$

Stable measures

Definition 13

A random variable X is called α-*stable*, $\alpha \in (0,2]$, if given two independent copies X_1, X_2 of X, for any $a, b > 0$ there exists a constant c such that

$$aX_1 + bX_2 \stackrel{d}{=} (a^\alpha + b^\alpha)^{1/\alpha} X + c$$

In particular the 2-stable random variables are the Gaussian random variables.

A measure is α-stable if its associated random variable is α-stable. If $c = 0$, we say the random variable (resp. measure) is symmetric α-stable.

From here onwards, we consider α-stable distributions represented as $X = \int_E z\Lambda(dz)$, where the family of Lévy measures are given in the form

$$\rho(z, du) = c_\alpha \left(\mathbf{1}_{u>0} + \mathbf{1}_{u<0}\right) |u|^{-1-\alpha}\, du, \qquad z \in E.$$

Mixing criteria for stable measures

Corollary 14 ([MP24])

Assume that the Banach space E is of stable type $\alpha \in (0,2)$, $\alpha \neq 1$, and let μ denote an α-stable distribution represented as $\int_E z\Lambda(dz)$. A bounded linear operator $T : E \to E$ that leaves μ invariant is strongly mixing with respect to μ if and only if for every $x^* \in E^*$ we have

$$\lim_{n\to\infty} \int_E |\operatorname{Re}\langle z, x^*\rangle|^{\alpha/2} |\operatorname{Re}\langle z, T^{*n}x^*\rangle|^{\alpha/2} \xi(dz) = 0$$

and

$$\lim_{n\to\infty} \int_E |\operatorname{Re}\langle z, x^*\rangle|^{\alpha/2} |\operatorname{Im}\langle z, T^{*n}x^*\rangle|^{\alpha/2} \xi(dz) = 0.$$

Rates of convergence

Definition 15

Let $T : E \to E$ be a bounded linear map such that μ is T-invariant. Define the quantity

$$I_n(f, g) := \int_E f(z)g(T^n z)\mu(dz) - \int_E f(z)\mu(dz) \int_E g(z)\mu(dz)$$

for each $f, g \in L^2(E, \mu)$ and $n \in \mathbb{N}$.

These quantities are related to the codifferences by

$$\begin{aligned}
&I_n(e^{i \operatorname{Re}\langle \cdot, x^* \rangle}, e^{-i \operatorname{Re}\langle \cdot, T^{*n} x^* \rangle}) \\
&= \left(\int_E e^{i \operatorname{Re}\langle z, x^* \rangle} \mu(dz) \int_E e^{-i \operatorname{Re}\langle z, x^* \rangle} \mu(dz) \right) \exp(C_\mu^{=}(x^*, T^{*n} x^*) - 1), \\
&I_n(e^{i \operatorname{Re}\langle \cdot, x^* \rangle}, e^{-i \operatorname{Im}\langle \cdot, T^{*n} x^* \rangle}) \\
&= \left(\int_E e^{i \operatorname{Re}\langle z, x^* \rangle} \mu(dz) \int_E e^{-i \operatorname{Im}\langle z, x^* \rangle} \mu(dz) \right) \exp(C_\mu^{\neq}(x^*, T^{*n} x^*) - 1).
\end{aligned}$$

Rate inequalities in the stable case

If μ is α-stable and $0 < \alpha < p \leq 2$, then

$$\left|C_\mu^{=}(x^*, T^{*n}x^*)\right| \leq \frac{16}{p^2} A_n \int_E |\operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle|^{p/2} \xi(dz) + 16\xi(E)B_n,$$

and

$$\left|C_\mu^{\neq}(x^*, T^{*n}x^*)\right| \leq \frac{16}{p^2} A_n \int_E |\operatorname{Re}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle|^{p/2} \xi(dz) + 16\xi(E)B_n,$$

where

$$A_n = \frac{2c_n^{p-\alpha}}{p-\alpha} \qquad \text{and} \qquad B_n = \frac{2}{\alpha c_n^\alpha},$$

for any non-negative sequence (c_n).

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Example: Bilateral forward shift

Given $\alpha \in (1,2)$ and $\alpha \leq p \leq 2$, consider a positive weight sequence (ω_n) such that

- there exist $\eta_1, \eta_2 \in (0,1)$ such that

$$\begin{cases} \omega_i \leq \eta_1, & i \geq 1, \\ \omega_i = 1, & i = -1, 0, \\ \omega_i \geq 1/\eta_2, & i \leq -2, \end{cases}$$

- $(\omega_{-i})_{i \geq 1}$ is strictly increasing, and
- if $p = \alpha$, then in addition $\inf_{i>0} \omega_i > 0$,

and define the weighted forward shift T on $\ell^p(\mathbb{Z})$ by $Te_n = \omega_{n+1} e_{n+1}$ for all n.

Example: Bilateral forward shift

Define the control measure

$$\xi(dz) = \frac{1}{4} \sum_{n=-\infty}^{\infty} k_n^{\alpha} \delta_{\{e_n, -e_n, ie_n, -ie_n\}}(dz),$$

where here δ is the Dirac delta function, and

$$k_n = \begin{cases} \prod_{0 \le i \le n} \omega_i, & n \ge 0, \\ \prod_{n < i \le 0} \frac{1}{\omega_i}, & n < 0, \end{cases}$$

and set $X = \int_E z\Lambda(dz)$ with distribution μ. It can be shown that X is well-defined in $\ell^p(\mathbb{Z})$.

Example: Bilateral forward shift

We have

$$\int_E |\operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle|^{p/2}\xi(dz) \leq \|x^*\|^p \sum_{i=-\infty}^{\infty} k_i^p \prod_{j=i+1}^{i+n} \omega_j^{p/2}.$$

Furthermore, direct estimates give:

- If $i \geq 0$ then $\displaystyle\sum_{i=0}^{\infty} k_i^p \prod_{j=i+1}^{i+n} \omega_j^{p/2} \leq \frac{\eta_1^{pn/2}}{1-\eta_1^p}$.
- If $i \leq -n$ then $\displaystyle\sum_{i=-\infty}^{n} k_i^p \prod_{j=i+1}^{i+n} \omega_j^{p/2} \leq \frac{\eta_2^{3pn/2}}{1-\eta_2^p}$.
- If $-n < i < 0$ then $\displaystyle\sum_{i=-n+1}^{-1} k_i^p \prod_{j=i+1}^{i+n} \omega_j^{p/2} \leq \left(\frac{\eta_2}{\eta_1}\right)^{p/2} \frac{\eta_1^{pn/2} - \eta_2^{p(n+1)/2}\eta_1^{-1/2}}{1-(\eta_2/\eta_1)}$.

It follows T is mixing with respect to μ.

Example: Bilateral forward shift

Furthermore, let $\eta := \max\{\eta_1, \eta_2\}$, and then

$$\int_E |\operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle|^{p/2}\xi(dz) = O(\eta^{pn/2}).$$

If $\alpha < p \leq 2$, then we can bound the convergence rates as

$$\left|C_\mu^{=}(x^*, T^{*n}x^*)\right| = O(c_n^{p-\alpha})O(\eta^{pn/2}) + O(c_n^{-\alpha})$$
$$\left|C_\mu^{\neq}(x^*, T^{*n}x^*)\right| = O(c_n^{p-\alpha})O(\eta^{pn/2}) + O(c_n^{-\alpha}).$$

In particular if $c_n = \eta^{-n/2}$ then the RHS is $O(\eta^{n\alpha/2})$.